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COMPLETE PERIODIC SYNCHRONIZATION OF DELAYED NEURAL NETWORKS WITH DISCONTINUOUS ACTIVATIONS

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Recently, the synchronization issue in chaotic systems has become a hot topic in nonlinear dynamics and has aroused great interest among researchers due to the theoretical significance and potential applications. In this paper, complete periodic synchronization is considered for the delayed neural networks with discontinuous activation functions. Under the framework of Filippov solution, a novel control method is presented by using differential inclusions theory, nonsmooth Lyapunov method and linear matrix inequality (LMI) approach. Based on a newly obtained necessary and sufficient condition, several criteria are derived to ensure the global asymptotical stability of the error system, and thus the response system synchronizes with the drive system. Moreover, the estimation gains are obtained. With these new and effective methods, complete synchronization is achieved. Simulation results are given to illustrate the theoretical results.

Keywords: Periodic synchronization; delayed neural networks; discontinuous activation functions; Filippov solution; nonsmooth analysis; differential inclusions; LMI approach.

1. Introduction

In mathematics and control theory, due to the conventional definition and simple existence conditions of solutions for differential equations, most theoretical results in analysis and control of dynamical systems are established under the assumptions of smoothness (continuously differentiable) condition of the given vector field. However, in various science and engineering applications, the system dynamics is discontinuous. Examples include dry friction, impacting machines, systems oscillating under the effect of an earthquake, power circuits, switching in electronic circuits and many others [Chen *et al.*, 2006; Cortés, 2008].

Recently, in the fields of signal processing, pattern recognition, parallel computation, complicated optimization problems, etc., neural network plays an increasingly important role, and its various properties have been systematically studied in [Cao, 2001; Huang *et al.*, 2005; Liang *et al.*, 2008; Song *et al.*, 2005]. However, there is little literature focusing on neural networks with discontinuous activation functions, though they frequently arise in practice [Forti & Nistri, 2003].

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From the theoretical point of view, the basic question is about the solution of the discontinuous dynamical systems. Does the classical definition of solutions still work for the discontinuous dynamical systems? How to ensure the existence and uniqueness of such solutions? It is well known that continuity of the vector field suffices to guarantee the existence of classical (continuously differentiable) solutions, as stated by Peanos theorem [Coddington & Levinson, 1955]. That is to say that the classical solutions might not exist, if the vector field is discontinuous. The existence of solutions for discontinuous dynamical systems is a delicate problem, as can be seen from the following example. Consider the differential equation [Cortés, 2008]

$$\dot{x} = f(x(t)), \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x(t)) = \begin{cases} -1, & x > 0, \\ 1, & x \leq 0, \end{cases} \tag{2}$$

which is discontinuous at zero. Suppose that there exists a continuously differentiable function $x : [0, t_1] \to \mathbb{R}$ such that $\dot{x}(t) = f(x(t))$ and $x(0) = 0$. Then $\dot{x}(0) = f(x(0)) = f(0) = 1$, which implies that, for all positive t sufficiently small, $x(t) > 0$ and hence $\dot{x}(t) = f(x(t)) = -1$, which contradicts the fact that $t \mapsto \dot{x}(t)$ is continuous. Hence, no classical solution starting from zero exists.

Actually, the discontinuous dynamical system has been an old research topic for decades. Several types of solutions have been available such as Caratheodory solutions, Filippov solutions, and sample-and-hold ones [Cortés, 2008]. Recently, much research interest has been focused on the notion of solutions with the Filippov framework [Filippov, 1988]. Such a notion has been utilized as a feasible approach in the field of mathematics and control for discontinuous dynamical systems. In 2003 and 2005, under the Filippov framework, sufficient conditions were obtained for the global asymptotical stability of the unique equilibrium point of discontinuous neural networks [Forti & Nistri, 2003; Forti *et al.*, 2005], which motivated the latter studies on neural networks with discontinuous activations [Huang & Cao, 2008; Huang *et al.*, 2009; Liu & Cao, 2009; Lu & Chen, 2006, 2008; Papini & Taddei, 2005].

On the other hand, the synchronization of two or more dynamical systems is a basis to understand an unknown dynamical system from one or more well-known dynamical systems. In other words, the response complexity of an unknown system to one or more well-known systems can be measured and compared through such synchronicity. Early studies about synchronization focused on dynamical behaviors in various periodic systems [Kuramoto, 1984; Winfree, 1980]. It is known that chaotic systems exhibit sensitive dependence on initial conditions. Just because of this property, it was long believed that chaotic systems defy synchronization. Until 1990, chaotic synchronization was first realized by Pecora and Carroll [1990]. Since then, chaotic synchronization has become a hot topic in nonlinear dynamics due to theoretical significance and potential applications. So far, many types of synchronization have been presented, such as identical or complete synchronization, generalized synchronization, phase synchronization, anticipated and lag synchronization (for details, see [Luo, 2009]). Synchronously, several control approaches have been developed to synchronize two or more dynamical systems such as drive-response, coupling control, adaptive control, feedback control, fuzzy control, observer-based control, manifold-based method and impulsive control, intermittent control, etc. It seems that these theories and methods have completely solved the problem of synchronization issue for any oscillator. However, still not much is known for complete synchronization of periodic oscillators. At a first glance, one might intuitively believe that since chaotic motion is more complicated than periodic motion, the synchronization of chaotic oscillators should also be more complicated than those of periodic oscillators. Nevertheless, this is not always the case and the synchronization patterns for periodic cases can actually be more complicated than those for chaotic cases, just as indicated in [Zou & Zhan, 2008a, 2008b], where an opposite result was given. Hence, it is our intention in this paper to tackle such an important yet challenging problem.

Motivated by the above discussions, in this paper, we will consider the complete periodic synchronization of delayed neural networks with discontinuous activations based on nonsmooth analysis and differential inclusions theory [Aubin & Cellina, 1984; Aubin & Frankowska, 1990]. Firstly, under the framework of Filippov, the existence of periodic solutions for such discontinuous neural systems can be guaranteed. Then, a controller is designed for the synchronization of periodic systems. Several sufficient criteria are derived to guarantee the global asymptotical stability of the error system. We note that, when proving the error system's global asymptotical stability, it is quite different from the classical Lyapunov methods. The Lyapunov functional is not smooth as usual any more due to the discontinuity of activations. Compared to previous literature, the main contributions of this paper are to initially formulate the periodic synchronization problem for discontinuous neural networks and to develop a delay-independent and LMI-based approach to solve it.

The rest of the paper is organized as follows. Section 2 gives some preliminaries. Section 3 discusses the existence of periodic solutions for the discontinuous neural networks. Section 4 presents some sufficient conditions for synchronization of the delayed drive and response system. In Sec. 5, simulation results aiming at substantiating the theoretical analysis are reported. The main conclusions are presented in Sec. 6.

Notations. The notations are quite standard. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript T denotes matrix transposition and the notation $X \geq Y$ (respectively $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively positive definite). I is the identity matrix with appropriate dimension. $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\|A\|_2$ its operator norm, i.e. $\|A\|_2 = \sup\{\|Ax\| : \|x\| = 1\} = \sqrt{\lambda_{\max} A^T A}$, where $\lambda_{\max}(\cdot)$ means the largest eigenvalue of A. $C([0,\omega],\mathbb{R}^n), L^1([0,\omega],\mathbb{R}^n)$ and $L^\infty([0,\omega],\mathbb{R}^n)$ are the spaces of continuous vector function, square integrable vector function and essentially bounded function on $[0,\omega]$, respectively. The notation $*$ always denotes the symmetric block in a symmetric matrix. Sometimes, the arguments of a function or a matrix will be omitted in the analysis when no confusion can arise.

2. Model Formulation and Preliminaries

In this paper, we consider the following neural networks described by the system

$$\dot{x} = -Dx(t) + Af(x(t)) + Bf(x(t-\tau)) + I(t), \tag{3}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is the state vector associated with the neurons; $D = \mathtt{diag}(d_1, d_2, \ldots, d_n)$ is an $n \times n$ constant diagonal matrix with $d_i > 0$, $i = 1, 2, \ldots, n$; $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively; $f(x) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))^T : \mathbb{R}^n \to \mathbb{R}^n$ is a diagonal mapping where $f_i, i = 1, 2, \ldots, n$, represents the neuron input–output activation; τ is a constant delay and $I(t) = (I_1(t), I_2(t), \ldots, I_n(t))^T$ a continuous ω-periodic function.

Throughout the paper, for each $i = 1, 2, \ldots, n$, it will be assumed that f_i is continuous in $\mathbb{R}$ except a finite number of points of discontinuity, ρ_k, where there exist finite right and left limits, $f_i(\rho_k^+)$ and $f_i(\rho_k^-)$, respectively, with $f_i(\rho_k^+) > f_i(\rho_k^-)$. Let $\mathcal{F}$ be the class of these functions.

In the following, we apply the framework of Filippov in discussing the solution of delayed neural networks (3).

Definition 1. Suppose $E \subset \mathbb{R}^n$. Then $x \mapsto F(x)$ is called as a set-valued map from $E \hookrightarrow \mathbb{R}^n$, if for each point x of a set $E \subset \mathbb{R}^n$, there corresponds a nonempty set $F(x) \subset \mathbb{R}^n$. A set-valued map F with nonempty values is said to be upper-semi-continuous at $x_0 \in E$ if, for any open set N containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M) \subset N$. $F(x)$ is said to have a closed (convex, compact) image if for each $x \in E$, $F(x)$ is closed (convex, compact).

Now we introduce the concept of Filippov solution. Consider the following system

$$\frac{dx}{dt} = f(x), \tag{4}$$

where $f(\cdot)$ is not continuous.

Definition 2. A set-valued map is defined as

$$F(x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} K[f(B(x,\delta)\backslash N)], \tag{5}$$

where $K(E)$ is the closure of the convex hull of set E, $B(x,\delta) = \{y : \|y - x\| \leq \delta\}$, and $\mu(N)$ is Lebesgue measure of set N. A solution in the sense of Filippov [1988] of the Cauchy problem for Eq. (4) with initial condition $x(0) = x_0$ is an absolutely continuous function $x(t), t \in [0,T]$, which satisfies $x(0) = x_0$ and differential inclusion:

$$\frac{dx}{dt} \in F(x), \quad \text{a.e. } t \in [0,T]. \tag{6}$$

Denoting

$$\mathbb{F}(x) \triangleq K[f(x)] = (K[f_1(x_1)], \ldots, K[f_n(x_n)]),$$

where $K[f_i(x_i)] = [f_i(x_i^-), f_i(x_i^+)],\ i = 1, \ldots, n$, we extend the concept of the Filippov solution to the differential Eq. (3) as follows:

Definition 3. A function $x : [-\tau, T) \to \mathbb{R}^n$, $T \in (0, +\infty]$, is a solution (in the sense of Filippov) of the discontinuous system (3) on $[-\tau, T)$, if:

(I) x is continuous on $[-\tau, T)$ and absolutely continuous on $[0, T)$;
(II) $x(t)$ satisfies

$$\begin{aligned}\dot{x}(t) \in -Dx(t) + AK[f(x(t))] \\ + BK[f(x(t-\tau))] + I(t) \\ \text{for a.e. } t \in [0, T). \qquad (7)\end{aligned}$$

Or equivalently,

(II′) there exists a measurable function $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T : [-\tau, T) \to \mathbb{R}^n$, such that $\alpha(t) \in K[f(x(t))]$ for a.e. $t \in [-\tau, T)$ and

$$\dot{x}(t) = -Dx(t) + A\alpha(t) + B\alpha(t-\tau) + I(t), \quad \text{for a.e. } t \in [0, T), \qquad (8)$$

where the single-valued function α is the so-called *measurable selection* of the function $\mathbb{F}$, which approximates $\mathbb{F}$ in some neighborhood of $Gragh(\mathbb{F})$.

It is obvious that the set-valued map $x(t) \hookrightarrow -Dx(t) + AK[f(x(t))] + BK[f(x(t-\tau))] + I(t)$ has nonempty compact convex values. Furthermore, it is upper-semi-continuous [Aubin & Cellina, 1984] and hence it is measurable. Here, we remark that all the set-valued functions obtained by Filippov regularization applied to functions $f \in \mathcal{F}$ verify the above several properties. Hence, by the measurable selection theorem [Aubin & Frankowska, 1990], if $x(t)$ is a solution of (3), then there exists a measurable function $\alpha(t) \in K[f(x(t))]$ such that for a.e. $t \in [0, +\infty)$, the Eq. (8) is true.

Definition 4. For any continuous function $\phi : [-\tau, 0] \to \mathbb{R}^n$ and any measurable function $\psi : [-\tau, 0] \to \mathbb{R}^n$, such that $\psi(s) \in K[f(\phi(s))]$ for a.e. $s \in [-\tau, 0]$, an absolute continuous function $x(t) = x(t, \phi, \psi)$ associated with a measurable function $\alpha(t)$ is said to be a solution of the Cauchy problem for system (3) on $[0, T)$ (T might be ∞) with initial value $(\phi(s), \psi(s))$, $s \in [-\tau, 0]$, if

$$\begin{cases} \dot{x}(t) = -Dx(t) + A\alpha(t) + B\alpha(t-\tau) + I(t), \\ \qquad\qquad \text{for a.e. } t \in [0, T), \\ x(s) = \phi(s), \quad \forall\, s \in [-\tau, 0], \\ \alpha(s) = \psi(s), \quad \text{for a.e. } s \in [-\tau, 0]. \end{cases} \qquad (9)$$

3. Existence of Periodic Solutions

In this section, we prove that under some conditions, system (9) has ω-periodic solutions. The following two lemmas should be recalled:

Lemma 1 [Douglas, 1972]. *Let X^* be dual space of Banach space X and S be closed unit ball of X^*. Then S is weakly-* compact.*

Note that the weak-* convergence is defined as: Let X be a Banach space and X^* its normed dual space, $\{f_n\} \subset X^*$ and $f \in X^*$, f_n is said to weakly-* converge to f in X^* (denote $w^* - \lim_{n\to\infty} f_n = f$), if for all $x \in X$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Lemma 2 [Dugundji & Granas, 1982]. *If X is a Banach space,*

$$2^X = \{\mathbb{C} : \mathbb{C} \subset X,\ \mathbb{C} \text{ is nonempty, compact and convex}\},$$

and $\mathbb{G} : X \to 2^X$ is an upper-semi-continuous set-valued map which maps bounded sets into relatively compact sets, then one of the following statements is true:

(a) *the set $\Gamma = \{x \in X : x \in \lambda\mathbb{G}(x), \lambda \in (0, 1)\}$ is unbounded;*
(b) *the $\mathbb{G}(\cdot)$ has a fixed point, i.e. there exists $x \in X$ such that $x \in \mathbb{G}(x)$.*

Theorem 1. *Suppose that $\mathbb{F}$ satisfies a growth condition (**g.c.**): there exist constants $K_1, K_2 \geq 0$ with*

$$\|\mathbb{F}(x)\| = \sup_{\xi \in \mathbb{F}(x)} \|\xi\| \leq K_1\|x\| + K_2. \qquad (10)$$

Then, there exists at least one periodic solution of system (3) in the sense of Eqs. (9).

Proof. Based on the detailed discussions in Sec. 2, the set-valued map $x(t) \hookrightarrow -Dx(t) + AK[f(x(t))] + Bf(x(t-\tau)) + I(t)$ is upper-semi-continuous with nonempty compact convex values, the local existence of a solution $x(t)$ of (9) can be guaranteed [Filippov, 1988]. In [Forti *et al.*, 2005], the solution's local existence was considered by step-by-step construction.

Denote $\bar{\theta} = \max_{1\leq i\leq n} \max_{-\tau\leq t\leq 0}\{\theta_i(t)\}$, $\bar{I} = \max_{1\leq i\leq n} \max_{-\tau\leq t\leq \infty}\{I_i(t)\}$. By (10), for a.e. $t \in [0, +\infty)$, one has

$$\begin{aligned}&\|-Dx(t) + AK[f(x(t))] + BK[f(x(t-\tau))] + I(t)\| \\ &\quad \leq \|D\|_2\|x(t)\| + \|A\|_2(K_1\|x(t)\| + K_2) \\ &\qquad + \|B\|_2(K_1\|x(t-\tau)\| + K_2) + \bar{I}\end{aligned}$$

$$
\begin{aligned}
&\leq \|D\|_2\|x(t)\| + \|A\|_2(K_1\|x(t)\| + K_2) \\
&\quad + \|B\|_2(K_1\bar{\theta} + K_1\|x(t)\| + K_2) + \bar{I} \\
&\leq (\|D\|_2 + K_1\|A\|_2 + K_1\|B\|_2)\|x(t)\| \\
&\quad + K_2\|A\|_2 + K_2\|B\|_2 + K_1\bar{\theta}\|B\|_2 + \bar{I} \\
&\triangleq \bar{A}\|x(t)\| + \bar{B}, \qquad (11)
\end{aligned}
$$

where $\bar{A} = \|D\|_2 + K_1(\|A\|_2 + \|B\|_2)$, $\bar{B} = K_2(\|A\|_2 + \|B\|_2) + \bar{\theta}K_1\|B\|_2 + \bar{I}$.

It follows that

$$\|x(t)\| \leq \|x(0)\| + \|\int_0^t \dot{x}(s)ds\|$$

$$\leq \|x(0)\| + \bar{B}t + \int_0^t \bar{A}\|x(s)\|ds.$$

By the Gronwall inequality, one has

$$\|x(t)\| \leq (\|x(0)\| + \bar{B}t)e^{\bar{A}t}.$$

Hence, since $x(t)$ remains bounded for positive times, it is defined on $[0, +\infty)$.

Next, we will show that the following differential inclusion with periodic boundary value conditions has a periodic solution

$$
\begin{cases}
\dot{x}(t) \in -Dx(t) + AK[f(x(t))] \\
\qquad + BK[f(x(t-\tau))] + I(t), \\
\qquad\qquad \text{for a.e. } t \in [0, T), \\
x(s) = \theta(s), \quad \forall\, s \in [-\tau, 0], \\
x(0) = x(\omega).
\end{cases} \qquad (12)
$$

For all $x \in C([0,\omega], \mathbb{R}^n)$, we denote $F(t, x(t)) = -Dx(t) + AK[f(x(t))] + BK[f(x(t-\tau))] + I(t)$ and $x(s) = \theta(s),\ \texttt{for}\ s \in [-\tau, 0]$.

Define a set-valued map δ, for all $x \in C([0,\omega]$,

$$
\begin{aligned}
\delta x := \{&y \in C[0,\omega] \mid y(0) = \theta(0), \\
&\dot{y}(t) \in -Dx(t) + AK[f(x(t))] \\
&+ BK[f(x(t-\tau))] + I(t), \\
&\text{a.e. } t \in [0,\omega] \text{ and } \dot{y}(t) \text{ is measurable}\}.
\end{aligned}
$$

By the measurable selection theorem [Aubin & Frankowska, 1990], there exists a measurable function $v(t) \in L^1[0,\omega]$ such that $v(t) \in F(t, x(t))$ for a.e. $t \in [0,\omega]$.

Let

$$y(s)|_{[-\tau,0]} = \theta(s),\ y(t) = \theta(0) + \int_0^t v(s)ds,$$

then $y \in \delta x$. Therefore, for all $x \in C[0,\omega]$, δx is nonempty and $\dot{y} \in L^1[0,\omega]$. So y is absolutely continuous.

Corresponding to Lemma 2, the remaining proof will be divided into three steps.

Step 1. δ maps bounded sets into relatively compact sets. In fact, let $\mathbb{D} \subset C[0,\omega]$ be bounded set. i.e. $\exists L > 0, \forall\, x \in \mathbb{D}, \|x\| \leq L$. Then, by (11), for $y \in \delta x, \dot{y} \in F(t, x(t))$, a.e. $t \in [0,\omega]$, one has

$$\|\dot{y}\| \leq \bar{A}L + \bar{B}$$

and

$$\|y\| \leq \|\theta(0)\| + \omega(\bar{A}L + \bar{B}).$$

Hence, $\delta(\mathbb{D}) = \{z | z = \delta x,\ \forall\, x \in \mathbb{D}\}$ is uniformly bounded and is obviously equi-continuous. By Arzela-Ascoli Theorem, $\delta(\mathbb{D})$ is relatively compact in $C[0,\omega]$.

Step 2. The set-valued map δ is upper-semi-continuous. From *Step* 1, we only need to prove that δ is closed. i.e. $\forall\ \bar{x}\ , x_n \in C[0,\omega]\ ,\ x_n \to \bar{x}\ ;\ y_n \in \delta x_n\ ,\ y_n \to \bar{y}$, we want to prove $\bar{y} \in \delta\bar{x}$. Choosing $\varepsilon = 1, \exists N > 0$, when $n \geq N$, one has

$$\|x_n(t)\| \leq \|\bar{x}(t)\| + 1.$$

Let $u_n(t) = \dot{y}_n(t)$, for a.e. $t \in [0,\omega]$, one gets

$$\|u_n(t)\| \leq \bar{B} + \bar{A}\|x_n(t)\| \leq \bar{B} + \bar{A}(\|\bar{x}\| + 1).$$

From Lemma 1, there exists a subsequence of u_n (still denotes u_n) weakly-* converging to $\bar{u}$. Specially, for all $\varphi \in L^\infty[0,\omega]$, one has

$$\int_0^\omega u_n\varphi dt \to \int_0^\omega \bar{u}\varphi dt$$

i.e. $\{u_n\}$ weakly converges to $\bar{u}$ in $L^1[0,\omega]$.

On the other hand,

$$
\begin{aligned}
y_n(t) &= \theta(0) + \int_0^t \dot{y}_n(s)ds \\
&= \theta(0) + \int_0^t u_n(s)ds \to \theta(0) + \int_0^t \bar{u}(s)ds.
\end{aligned}
$$

So,

$$\bar{y}(t) = \theta(0) + \int_0^t \bar{u}(s)ds.$$

By the convergence theorem [Aubin & Cellina, 1984], one has

$$\dot{\bar{y}}(t) = \bar{u}(t) \in F(t, \bar{x}), \quad \text{for a.e. } t \in [0,\omega].$$

This implies that $\bar{y} \in \delta\bar{x}$.

Step 3. Now, we will prove that the set

$$\{x \in C[0,\omega] \mid x \in \lambda\delta x, \lambda \in (0,1)\}$$

is bounded.

For all $0 < \lambda < 1$, $x \in \lambda\delta x$, there exists $y \in \delta x$ such that $x = \lambda y$. Then for a.e. $t \in [0,\omega]$, one has

$$\|\dot{y}(t)\| \le \bar{B} + \bar{A}\|x\|.$$

Thus, for a.e. $t \in [0,\omega]$,

$$\begin{aligned}\|y(t)\| &\le \|\theta(0)\| + \int_0^t (\bar{B} + \bar{A}\|x(s)\|)ds \\ &\le \|\theta_0\| + \omega\bar{B} + \bar{A}\int_0^t \|x(s)\|ds.\end{aligned}$$

Denote

$$\psi(t) = \|\theta(0)\| + \omega\bar{B} + \bar{A}\int_0^t \|x(s)\|ds.$$

Then,

$$\dot{\psi}(t) = \bar{A}\|x(t)\| = \lambda\bar{A}\|y(t)\| \le \bar{A}\|y(t)\| \le \bar{A}\psi(t),$$

and

$$\psi(t) \le \psi(0)e^{t\bar{A}} \le (\|\theta(0)\| + \omega\bar{B})e^{\omega\bar{A}}.$$

Hence,

$$\|x(t)\| \le \|y(t)\| \le \psi(t) \le (\|\theta(0)\| + \omega\bar{B})e^{\omega\bar{A}}.$$

By Lemma 2, δ has a fixed point x^*, which obviously shows that x^* is a solution of system (12), i.e. the neural network (3) has an ω-periodic solution in the sense of Eq. (9). ■

In this paper, we consider model (3) as the master system. The response system is

$$\begin{aligned}\dot{y} = -Dy(t) + Af(y(t)) + Bf(y(t-\tau)) \\ + I(t) + u(t),\end{aligned} \tag{13}$$

where D, A, B are matrices which are the same as in (3), $u(t)$ is the controller. It has the same structure as the drive system.

Let error state be $e(t) = y(t) - x(t)$, subtracting (3) from (13), yields the synchronization error dynamical system as follows:

$$\dot{e}(t) = -De(t) + Ag(t) + Bg(t-\tau) + u(t), \tag{14}$$

where $g(t) = f(y(t)) - f(x(t))$ satisfying $g(t) = 0 \Leftrightarrow e(t) = 0$.

In many real applications, we are interested in designing a memoryless feedback controller

$$u(t) = Ge(t), \tag{15}$$

where $G \in \mathbb{R}^{n\times n}$ is a constant gain matrix. For a special case where the information on the size of delay τ is available, we also consider a delayed feedback controller of the following form:

$$u(t) = G_1 e(t) + G_2 e(t-\tau), \tag{16}$$

Although a memoryless controller (15) has an advantage of easy implementation, its performance cannot be better than a delayed feedback controller (16) which utilizes the available information of the size of delay.

Let $u(t) = G_1 e(t) + G_2 e(t-\tau)$, and substituting it into (13), one obtains

$$\begin{aligned}\dot{e} = (-D + G_1)e(t) + G_2 e(t-\tau) \\ + Ag(t) + Bg(t-\tau).\end{aligned} \tag{17}$$

4. Synchronization Criteria

In this section, we will prove that the response system (13) synchronizes with the drive system (3). Obviously, we can solve the problem by proving the global asymptotical stability of error system defined by (17).

Due to the discontinuity of f and then g, by Definition 3, there exists $\gamma(t) \in K[g(t)]$ such that $0 \in K[g(0)]$ and

$$\begin{aligned}\dot{e} = (-D + G_1)e(t) + G_2 e(t-\tau) \\ + A\gamma(t) + B\gamma(t-\tau).\end{aligned} \tag{18}$$

The following two definitions and one lemma should be recalled which will be utilized in our main results.

Definition 5 [Clarke, 1983]. Given $f : \mathbb{R}^n \to \mathbb{R}$, the right directional derivative of f at x in the direction $v \in \mathbb{R}^n$ is defined as

$$f'(x; v) = \lim_{h\to 0^+} \frac{f(x+hv) - f(x)}{h},$$

when this limit exists. The generalized directional derivative of f at x in the direction $v \in \mathbb{R}^n$ is defined as

$$\begin{aligned}f^0(x; v) &= \limsup_{\substack{y\to x\\ h\to 0^+}} \frac{f(y+hv) - f(y)}{h} \\ &= \lim_{\substack{\delta\to 0^+\\ \varepsilon\to 0^+}} \sup_{\substack{y\in B(x,\delta)\\ h\in[0,\varepsilon)}} \frac{f(y+hv) - f(y)}{h}.\end{aligned}$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is regular at $x \in \mathbb{R}^n$ if, for all $v \in \mathbb{R}^n$, the right directional derivative of f at x in the direction of v exists, and $f'(x; v) = f^0(x; v)$.

Note that regular functions admit the directional derivative for all directions $v \in \mathbb{R}^n$, although the derivative may be different for different directions. A function that is continuously differentiable at x is regular at x. A useful property is that a locally Lipschitz and convex function is regular.

Definition 6. Function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is C-regular, if $V(x)$ is:

(i) regular in $\mathbb{R}^n$;
(ii) positive definite, i.e. $V(x) > 0$ for $x \neq 0$, and $V(0) = 0$;
(iii) radially unbounded, i.e. $V(x) \to +\infty$ as $\|x\| \to +\infty$.

Note that a C-regular Lyapunov function V is not necessarily differentiable. Suppose that $x(t) : [0, +\infty) \to \mathbb{R}^n$ is absolutely continuous on any compact interval of $[0, +\infty)$. The next lemma gives a chain rule for computing the time derivative of the composed function $V(x(t)) : [0, +\infty) \to \mathbb{R}$.

Lemma 3 [Clarke, 1983]. *If $V(x) : \mathbb{R}^n \to \mathbb{R}$ is C-regular and $x(t) : [0, +\infty) \to \mathbb{R}^n$ is absolutely continuous on any compact interval of $[0, +\infty)$, then $x(t)$ and $V(x(t)) : [0, +\infty) \to \mathbb{R}$ are differential for* a.e. $t \in [0, +\infty)$, *and one has*

$$\frac{d}{dt}V(x(t)) = \left\langle \varsigma, \frac{dx}{dt} \right\rangle, \quad \forall \varsigma \in \partial V(x(t)).$$

Now, a new necessary and sufficient condition will be given, which is essential for establishing the synchronization criterion for (13) and (3).

Lemma 4. *There exists a matrix X such that*

$$\begin{pmatrix} P & Q & T & X \\ Q^T & R & 0 & V \\ T^T & 0 & N & L \\ X^T & V^T & L^T & S \end{pmatrix} < 0, \tag{19}$$

if and only if

$$\begin{pmatrix} P & Q & T \\ Q^T & R & 0 \\ T^T & 0 & N \end{pmatrix} < 0, \tag{20}$$

and

$$\begin{pmatrix} R & 0 & V \\ 0 & N & L \\ V^T & L^T & S \end{pmatrix} < 0. \tag{21}$$

Proof. Necessity is obvious since the left-hand side of both (20) and (21) are submatrices in the principal diagonal of the left-hand side of (19). For sufficiency, left-multiply (19) by

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & -V^TR^{-1} & -L^TN^{-1} & I \end{pmatrix}$$

and right-multiply its transpose, it is seen that (19) is equivalent to

$$\begin{pmatrix} P & Q & T & X - QR^{-1}V - TN^{-1}L \\ Q^T & R & 0 & 0 \\ T^T & 0 & N & 0 \\ X^T - V^TR^{-1}Q^T - L^TN^{-1}T^T & V^T & L^T & S - V^TR^{-1}V - L^TN^{-1}L \end{pmatrix} < 0.$$

But the above is clearly satisfied for

$$X = QR^{-1}V + TN^{-1}L$$

if (20) and (21) are satisfied in view of the famous Shur's complement.

Before proceeding to the main results, we further assume the set-valued map $\mathbb{F}$ satisfy:

(**L**.) For each $i \in 1, 2, \ldots, n$, there exists a constant L_i such that for any $\mu \neq \nu \in \mathbb{R}$, for all $\zeta_i \in K[f_i(\mu)]$, $\eta_i \in K[f_i(\nu)]$,

$$\frac{\zeta_i - \eta_i}{\mu - \nu} > L_i, \tag{22}$$

where the constant $L_i \in \mathbb{R}$ may be positive, negative or zero.

Theorem 2. *Let $\mathbb{F}$ satisfy (g.c.) and (L.), for given estimate gain matrices G_1, G_2, the response system (13) globally synchronizes with the drive system (3) if there exist two diagonal matrices $\check{P} = \mathrm{diag}(\check{p}_1, \check{p}_2, \ldots, \check{p}_n)$, $\hat{P} = \mathrm{diag}(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)$ with $\hat{p}_i > 0, i = 1, 2, \ldots, n$, two positive definite matrices $Q = (q_{ij})_{n\times n}$ and $R = (r_{ij})_{n\times n}$ such that the following LMIs hold*:

$$\begin{bmatrix} -2\check{P}D + \check{P}G_1 + G_1^T\check{P} + R & \check{P}G_2 & \check{P}B \\ * & -R & 0 \\ * & * & -Q \end{bmatrix} < 0, \tag{23}$$

$$\begin{bmatrix} -R & 0 & G_2^T\hat{P} \\ * & -Q & B^T\hat{P} \\ * & * & \hat{P}A + A^T\hat{P} + Q \end{bmatrix} < 0. \tag{24}$$

Proof. By (22) and the density of real numbers, there exist two arrays of constants $\check{p}_i$ and $\hat{p}_i$ such that

$$\frac{\zeta_i - \eta_i}{\mu - \nu} > -\frac{\check{p}_i}{\hat{p}_i}, \tag{25}$$

where $\hat{p}_i > 0, i = 1, 2, \ldots, n$.

Consider the following Lyapunov functional

$$V(t) = e^T(t)\check{P}e(t) + 2\sum_{i=1}^{n}\hat{p}_i\int_0^{e_i(t)} g_i(s)ds + \int_{t-\tau}^{t}\gamma^T(s)Q\gamma(s)ds + \int_{t-\tau}^{t} e^T(s)Re(s)ds. \tag{26}$$

From (25), there exists a small constant $\varepsilon > 0$ such that

$$\frac{\zeta_i - \eta_i}{\mu - \nu} \geq \frac{-\check{p}_i + \varepsilon}{\hat{p}_i}. \tag{27}$$

It follows that

$$\check{p}_i e_i^2(t) + 2\hat{p}_i\int_0^{e_i(t)} g_i(s)ds \geq \varepsilon e_i^2(t),$$

and thus

$$e^T(t)\check{P}e(t) + 2\sum_{i=1}^{n}\hat{p}_i\int_0^{e_i(t)} g_i(s)ds \geq \varepsilon\|e(t)\|^2.$$

By Lemma 3, one obtains

$$\begin{aligned}\frac{dV(t)}{dt} &= e^T(t)[2\check{P}(-D + G_1) + R]e(t) \\ &\quad + 2e^T(t)\check{P}G_2e(t-\tau) + 2e^T(t)[\check{P}A - D\hat{P} \\ &\quad + G_1^T\hat{P}]\gamma(t) + 2e^T(t)\check{P}B\gamma(t-\tau) \\ &\quad - e^T(t-\tau)\mathrm{Re}(t-\tau) + 2e^T(t-\tau)G_2^T\hat{P}\gamma(t) \\ &\quad + \gamma^T(t)[2\hat{P}A + Q]\gamma(t) \\ &\quad + 2\gamma^T(t)\hat{P}B\gamma(t-\tau) - \gamma^T(t-\tau)Q\gamma(t-\tau) \\ &\leq [e^T(t), e^T(t-\tau), \gamma^T(t-\tau), \gamma^T(t)]\Omega \\ &\quad \times \begin{bmatrix} e(t) \\ e(t-\tau) \\ \gamma(t-\tau) \\ \gamma(t) \end{bmatrix},\end{aligned} \tag{28}$$

where

$$\Omega = \begin{pmatrix} -2\check{P}D + \check{P}G_1 + G_1^T\check{P} + R & \check{P}G_2 & \check{P}B & \check{P}A - D\hat{P} + G_1^T\hat{P} \\ * & -R & 0 & G_2\hat{P} \\ * & * & -Q & B^T\hat{P} \\ * & * & * & \hat{P}A + A^T\hat{P} + Q \end{pmatrix}.$$

By Lemma 4, (23) and (24) are equivalent to $\Omega < 0$. Therefore, the solution $e(t) = 0$ of (18) is globally asymptotically stable, so the response system (13) globally synchronizes with the drive system (3). This completes the proof. ■

Remark 1. Due to the discontinuity of activation f, so does the function g, the Lyapunov–Krasovskii functional $V(t)$ is not differential in $[0, +\infty)$. But $V(t)$ is a locally Lipschitz and convex function on $\mathbb{R}^n$, hence it is regular in $\mathbb{R}^n$ and then is C-regular (according to Definition 6). Hence, we can utilize Lemma 3 to calculate the derivative of $V(t)$, among which, (18) is utilized repeatedly (see [Clarke, 1983], for details).

Remark 2. Previously, most theoretical results about the stability or synchronization for neural networks are always established under the assumption of smoothness or global Lipshitzian [Cao, 2001; Huang *et al.*, 2005; Liang *et al.*, 2008], where the Lipshitzian constant plays a very important role in magnifying the derivative of Lyapunov functional, and then can make the obtained LMI more solvable. Here, in this paper, we could not depend on the Lipshitzian constant due to the discontinuity of activations, so it seems more difficult when solving the LMIs (23) and (24).

In order to show the design of estimate gain matrix G_1 and G_2, a proper transformation is made to obtain the following theorem:

Theorem 3. *Let $\mathbb{F}$ satisfy (g.c.) and (L.), for the given constants m_1 and m_2, the response* *system (13) globally synchronizes with the drive system (3) if there exist one diagonal matrix* $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$ *with* $m_2 p_i > 0, i = 1, 2, \ldots, n$, *two positive definite matrices* $Q = (q_{ij})_{n\times n}$ *and* $R = (r_{ij})_{n\times n}$, *two arbitrary matrices* G'_1 *and* G'_2 *such that the following LMIs hold*:

$$\begin{bmatrix} -2m_1PD + m_1G'_1 + m_1G'^T_1 + R & m_1G'_2 & m_1PB \\ * & -R & 0 \\ * & * & -Q \end{bmatrix} < 0, \tag{29}$$

$$\begin{bmatrix} -R & 0 & m_2G'^T_2 \\ * & -Q & m_2B^TP \\ * & * & m_2PA + m_2A^TP + Q \end{bmatrix} < 0. \tag{30}$$

Moreover, the estimation gain $G_1 = P^{-1}G'_1$ and $G_2 = P^{-1}G'_2$.

Proof. Let $\check{P} = m_1P$, $\hat{P} = m_2P$ and $G'_1 = PG_1$, $G'_2 = PG_2$ in Theorem 2, it is obvious to see. ■

Remark 3. It is well known that the asymptotical or exponential stability of delayed neural networks [Cao, 2001; Huang *et al.*, 2005] has been a focal subject for research due to their great practical value and wide applications. And some stability issues for discontinuous neural networks are also being researched [Cortés, 2008; Forti & Nistri, 2003; Forti *et al.*, 2005; Huang & Cao, 2008; Huang *et al.*, 2009; Liu & Cao, 2009; Lu & Chen, 2006, 2008; Papini & Taddei, 2005]. However, there are few works on the synchronization of discontinuous delayed neural networks. In this paper, based on the concept of Filippov solution, the periodic synchronization control problem has been considered for delayed neural networks with discontinuous activations.

5. Illustrative Examples

Example 1. Consider the second-order drive system of a discontinuous delayed neural network as follows:

$$\dot{x} = -Dx(t) + Af(x(t)) + Bf(x(t-\tau)) + I(t), \tag{31}$$

and the response system is given as follows:

$$\begin{aligned} \dot{y} = -Dy(t) + Af(y(t)) + Bf(y(t-\tau)) \\ + I(t) + u(t), \end{aligned} \tag{32}$$

with

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad A = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix},$$

$$B = \begin{bmatrix} -1 & 0.9 \\ 1 & -2 \end{bmatrix},$$

$$I(t) = \begin{bmatrix} -25\cos(2t) \\ 25\sin(2t) \end{bmatrix}, \quad \tau = 0.1,$$

where the activation function is defined as $f(s) = s + \mathtt{sign}(s)$; $x(t) = [x_1(t), x_2(t)]^T$, $y(t) = [y_1(t), y_2(t)]^T$ are the state vectors. Let $m_1 = m_2 = 2$, from Theorem 3 and employing LMI toolbox in Matlab, we can obtain the following feasible solutions:

$$P = \begin{bmatrix} 3.8493 & 0 \\ 0 & 3.8493 \end{bmatrix}, \quad Q = \begin{bmatrix} 35.1632 & 0 \\ 0 & 35.1632 \end{bmatrix},$$

$$R = \begin{bmatrix} 39.5322 & 0 \\ 0 & 39.5322 \end{bmatrix},$$

$$G'_1 = \begin{bmatrix} -8.2182 & 0 \\ 0 & -8.2182 \end{bmatrix}, \quad G'_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We can see that the estimation gain $G_2 = P^{-1}G'_2 = 0$, which means the delayed feedback controller (16) degenerates to the memoryless feedback controller (15). Certainly, the memoryless feedback controller (15) could be used to stabilize system (3). However, there has been extensive interest in studying the effect of time delay on the feedback systems because time delay is ubiquitous in most physical, chemical, biological, neural and other natural systems due to finite propagation speeds of signals, finite processing times in synapses and finite reaction times. Therefore, we need to consider delay-dependent feedback controller (16). Although both the memoryless feedback controller (15) and the delay-dependent feedback controller (16) could be used to stabilize system (3), delay is ubiquitous in the real system and (16) is a more general controller. Thus in this example, without loss of generality, we

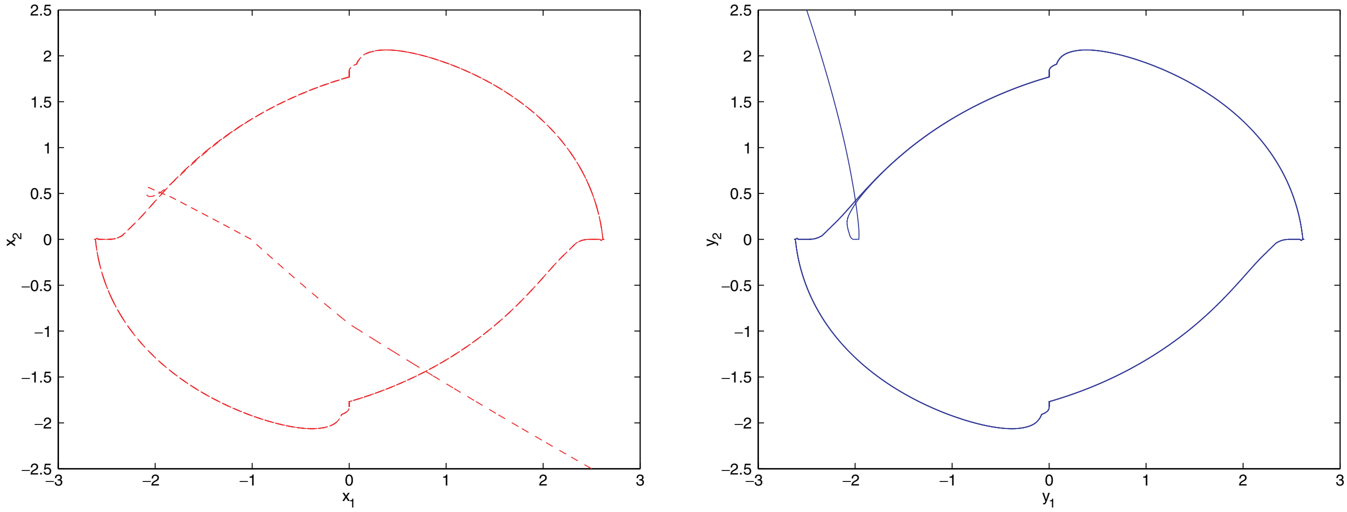

Fig. 1. Phase trajectories of drive (left) and response (right) systems for Example 1.

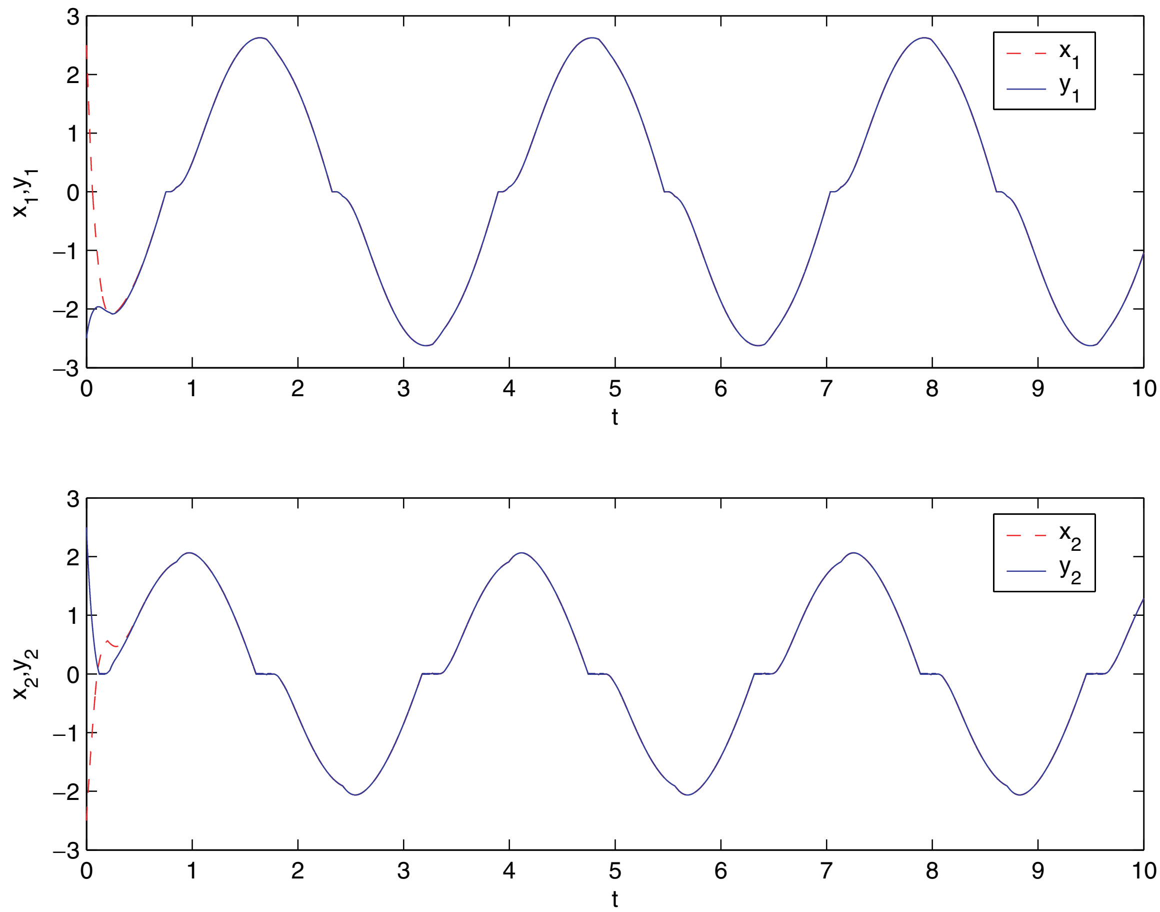

Fig. 2. Time evolution of each variable of coupled neural networks for Example 1.

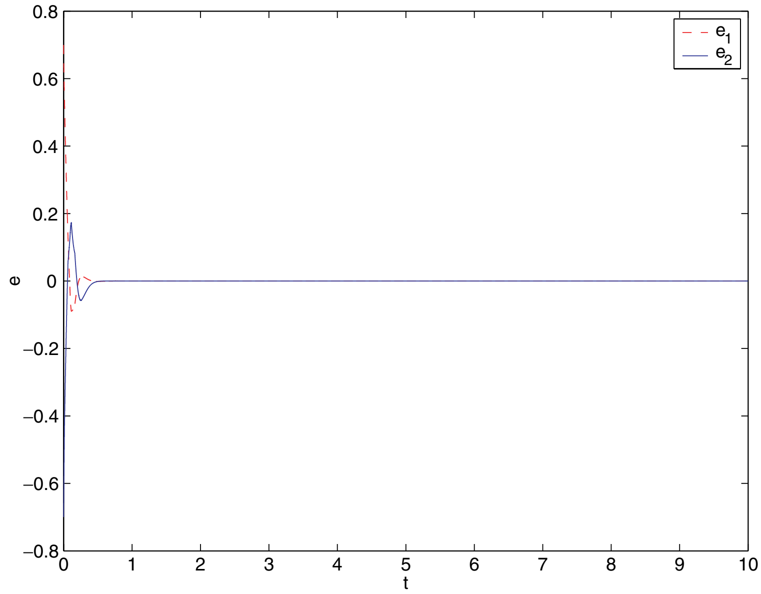

Fig. 3. The synchronization error of the state variables $e(t) = y(t) - x(t)$ for Example 1.

can further assume that G_2' in Theorem 3 is a positive definite matrix. Using LMI toolbox again, we can reobtain the following feasible solutions:

$$P = \begin{bmatrix} 3.8347 & 0 \\ 0 & 3.8347 \end{bmatrix},$$

$$Q = \begin{bmatrix} 35.4376 & 0 \\ 0 & 35.4376 \end{bmatrix},$$

$$R = \begin{bmatrix} 39.7899 & 0 \\ 0 & 39.7899 \end{bmatrix},$$

$$G_1' = \begin{bmatrix} -8.3910 & 0 \\ 0 & -8.3910 \end{bmatrix},$$

$$G_2' = \begin{bmatrix} 0.8161 & 0 \\ 0 & 0.8161 \end{bmatrix},$$

and then get the estimation gains:

$$G_1 = P^{-1}G_1' = \begin{bmatrix} -2.1882 & 0 \\ 0 & -2.1882 \end{bmatrix},$$

$$G_2 = P^{-1}G_2' = \begin{bmatrix} 0.2128 & 0 \\ 0 & 0.2128 \end{bmatrix}.$$

The drive system (31) is a periodic oscillation as shown in the left of Fig. 1. From Theorem 3, the delayed feedback controller can be designed as $u(t) = G_1e(t) + G_2e(t - \tau)$, and the response

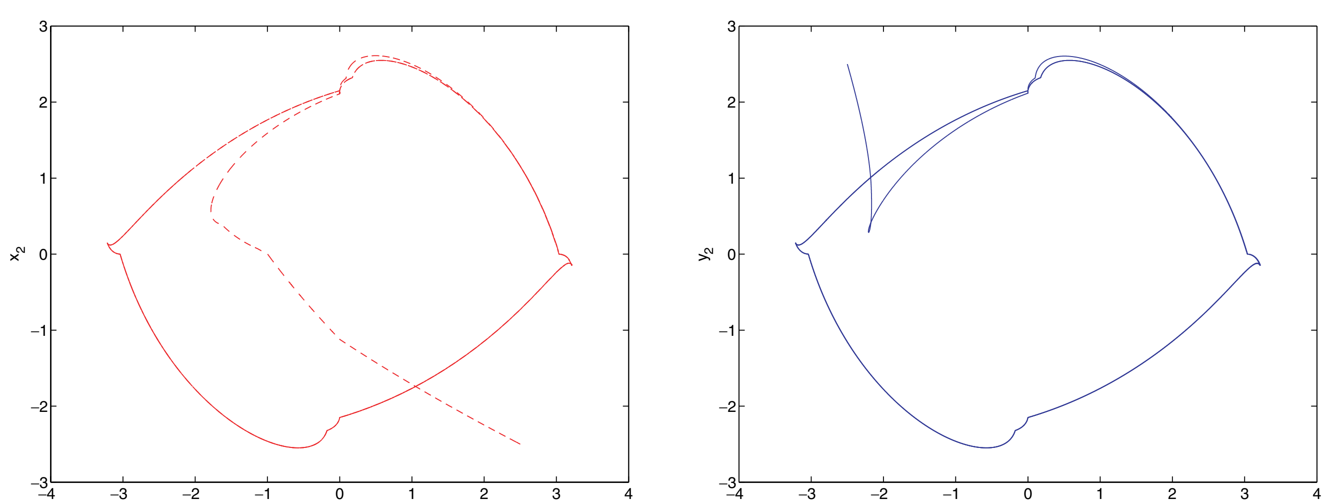

Fig. 4. Phase trajectories of drive (left) and response (right) systems for Example 2.

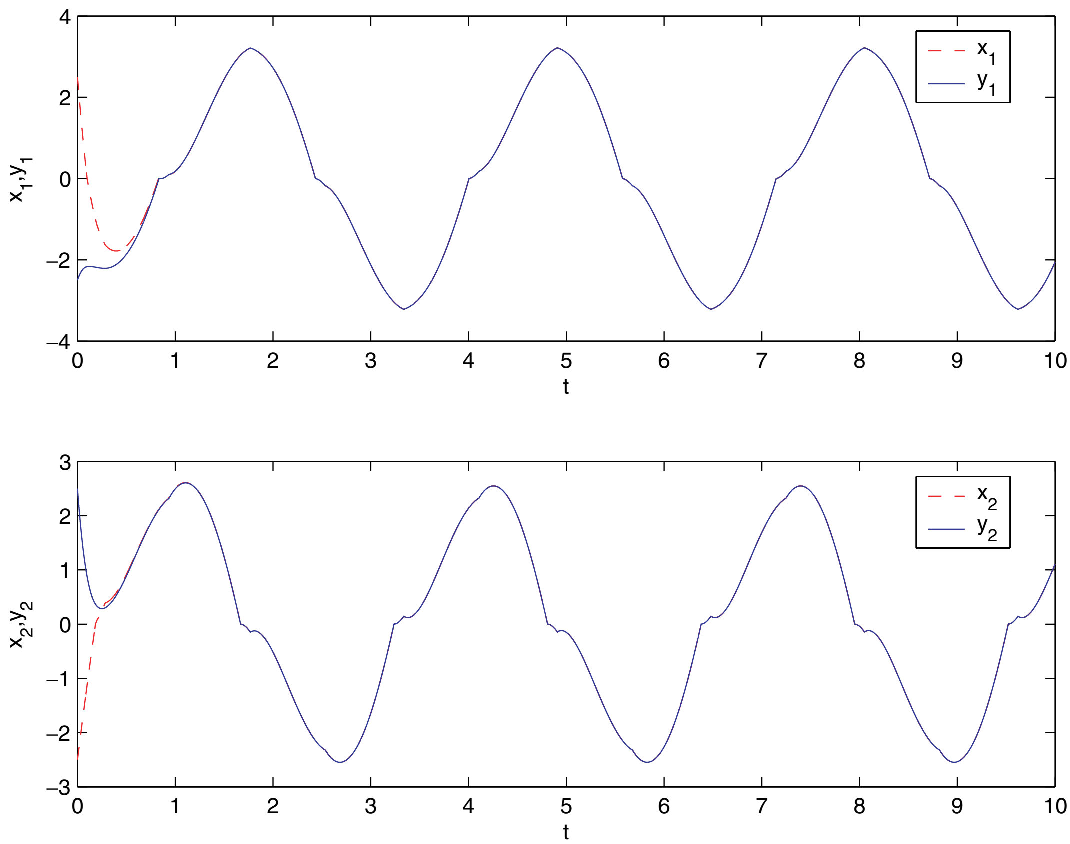

Fig. 5. Time evolution of each variable of coupled neural networks for Example 2.

system (32) is shown to the right of Fig. 1. Figure 2 shows the temporal evolution of each variable of the coupled neural networks $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$. Figure 3 depicts the synchronization error of the state variables between the drive system and the response system. It is clearly seen in Figs. 2 and 3 that the response system (32) synchronizes with the drive system (31).

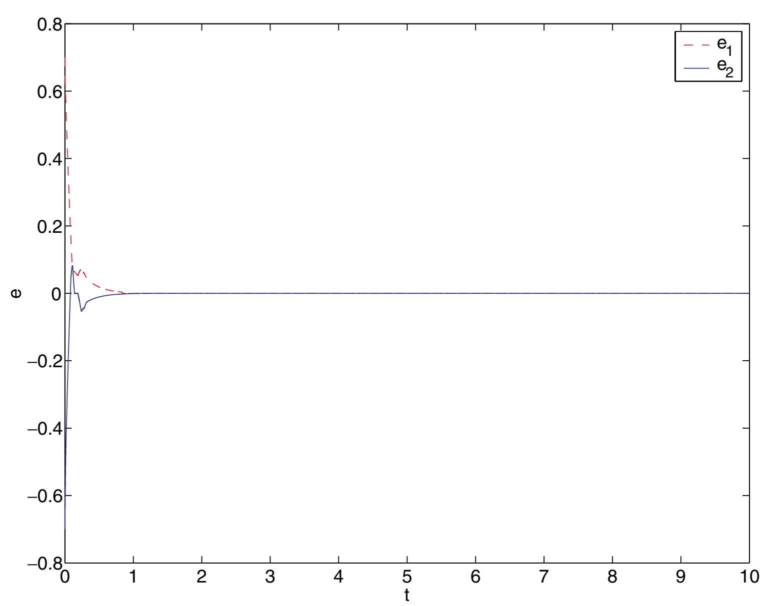

Fig. 6. The synchronization error of the state variables $e(t) = y(t) - x(t)$ for Example 2.

Example 2. We still consider the two-order neural network (31) as the drive system and (32) as the corresponding response system but the activation functions are changed to be discontinuous bounded ones defined as $f(s) = \mathtt{sign}(s)$ and the input $I(t) = [-16\cos(2t), 16\sin(2t)]^T$.

Similar to Example 1, by Theorem 3 the response system (32) can synchronize with the drive system (31). Simulation results are depicted in Figs. 4–6.

6. Conclusions

In recent years, the theory about the discontinuous neural networks and its application in practice is building up. In this paper, based on the concept of Filippov solution, we have considered complete periodic synchronization of discontinuous delayed neural networks. By using the differential inclusions theory and LMI method, several new sufficient conditions ensuring the global asymptotical stability for the error system have been derived. Meanwhile, the estimation gains can be obtained. The obtained results are novel since there are few works on the periodic complete synchronization of delayed discontinuous system. Finally, two numerical examples have been provided to illustrate the usefulness of our results.

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