

Delay-dependent multistability in recurrent neural networks[☆]

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ABSTRACT

In this article, we focus on the delay-dependent multistability in recurrent neural networks. By constructing Lyapunov functional and using matrix inequality techniques, a novel delay-dependent multistability criterion is derived. The obtained results are more flexible and less conservative than previously known criteria. Two examples are given to show the effectiveness of the obtained criteria. Furthermore, some interesting delay-dependent dynamic behaviors have been showed in a special case, for example, we find that there is the coexistence of stable equilibria and stable limit cycles in the single neuron. Also, when the neurons are coupled, then the stable patterns are more complex.

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1. Introduction

In recent years, neural networks have attracted more and more attention of researchers. Ranging from signal processing, pattern recognition, programming problems and static image processing, neural networks have witnessed a large amount of successful applications in many fields (see Cichocki, 2002; Cichocki & Unbehauen, 1993; Forti, Nistri, & Quincampoix, 2004; Forti & Tesi, 1995; Karhunen, Hyvarinen, Vigarino, Hurri, & Oja, 1997; Xia, Leung, & Bosse, 2002). And, these applications depend heavily on the network's dynamics. As practical applications of neural networks, multistability is a necessary feature for associative memory storage and pattern recognition. Multistability describes the coexistence of multiple stable patterns (Chua, 1998; Foss, Longtin, Mensour, & Milton, 1996; Hopfield, 1984; Morita, 1993), including stable equilibria and stable limit cycles. In Cohen (1992), two distinct but related constructive methods are provided for constructing systems of ordinary differential equations with arbitrary numbers of stable patterns.

Multistability in the delayed neural networks:

$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^n \alpha_{ij} g_j(x_j(t - \tau_{ij})) + J_i, \quad i = 1, 2, \dots, n \quad (1)$$

is discussed by Cheng, Lin, and Shih (2006). It is found that an n -neuron cellular neural networks can have up to 2^n locally stable equilibria. And, Cheng, Lin, and Shih (2007) has studied a general delayed neural networks:

$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^n \alpha_{ij} g_j(x_j(t)) + \sum_{j=1}^n \beta_{ij} g_j(x_j(t - \tau_{ij})) + J_i, \quad i = 1, 2, \dots, n. \quad (2)$$

In addition, the multistability of cellular neural networks with and without delays is investigated by Zeng, Wang, and Liao (2004) and Zeng, Huang, and Wang (2005). Furthermore, Zeng and Wang (2006) and Cheng et al. (2007) investigated the conditions for the existence of multiple stable periodic orbits evoked by periodic external inputs, in which all the stable patterns are limit cycles with the same periodic time. And, if orbits converge to the same periodic orbits, then they will be synchronized.

Since time delays are often encountered due to measurement and computational delays, which may result in oscillation and instability, the stability and multistability analyses of delayed neural networks have received considerable attention (e.g., see Cao, Ho, & Huang, 2007; Cao & Li, 2005; He & Wu, 2006; Li & Chen, 2007; Lou & Cui, 2006; Singh, 2006; Wang, Shu, Liu, Ho, & Liu, 2006; Xu & Lam, 2006). However, most investigations on multistability have focused on the delay-independent stability analysis. In general, the delay-dependent stability criteria are less conservative than delay-independent ones. Though Cheng et al. (2007) obtained a delay-dependent multistability criterion using their theory of quasi-convergence, there was a strong possibility that their criterion may be more conservative since they result from the strongly order preserving of the semiflow generated by the solution of neural networks (2). To overcome this conservatism, we shall derive

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a new delay-dependent multistability criterion for the neural networks by utilizing Linear Matrix Inequality (LMI) convex optimization approach.

Recently, the LMI-based techniques have been successfully used to tackle various stability problems for neural networks with or without delay (see Cao et al., 2007; Cao & Li, 2005; Lou & Cui, 2006; Wang et al., 2006; Xu & Lam, 2006). The main advantage of the LMI-based approaches is that the LMI stability conditions can be solved numerically using the effective interior-point algorithm. And, the delay-dependent stability is considered for neural networks based on LMI approach in Xu and Lam (2006) and Wang et al. (2006). Beside the stability problems, the LMI approaches (e.g. Lu & Chen, 2004; Yu & Cao, 2007) have also been used successfully to synchronize and estimate the state of the respective neural networks (see He, Wang, Wu, & Lin, 2006; Wang, Ho, & Liu, 2005).

Furthermore, to learn how delay effect on the multistability of the neural network, a special case will be investigated. Associating with the multistability criteria derived in this article and numerical simulations, we shall explore an interesting phenomenon that there coexist two stable equilibrium points and one stable limit cycle in a single neuron. It is different from the Hopf bifurcation (see Hassard, Kazarinoff, & Wan, 1981; Song, Han, & Wei, 2005; Zhu & Huang, 2007) and the other coexisting phenomenon for stable patterns (e.g. Campbell, Ncube, & Wu, 2006). Consider the neural network coupled by two neurons with small connection strength. Besides the compound stable patterns from the existing stable patterns of the single neuron, some new stable patterns emerge. There exist several types of stable patterns, which contain both stable equilibrium points and stable limit cycles with different periodic times. Different from the stable periodic orbits evoked by periodic external inputs, the orbits, which converge to the same stable pattern, can be asynchronous.

The rest of the article is organized as follows. In Section 2, the existence of multiple equilibria is introduced. In Section 3, the delay-dependent multistability criteria are derived. And, two numerical examples are illustrated in Section 4. In Section 5, we give an example to show the coexistence of different types of stable patterns. Finally, the conclusions are given in Section 6.

2. Multiple equilibria

Consider the neural network with delay as follows,

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij} g(x_j(t)) + \sum_{j=1}^n b_{ij} g(x_j(t - \tau)) + I_i, \quad i = 1, 2, \dots, n. \quad (3)$$

And, we assume the activation functions $g(x)$ have the sigmoidal configuration, which satisfies the following properties:

$$g \in \mathcal{C}^2, \quad \begin{cases} \mu_i^- < g(\xi) < \mu_i^+, & \dot{g}(\xi) > 0, \\ (\xi - \zeta_i) \dot{g}(\xi) < 0 & \text{for all } \xi \in \mathbb{R}, \\ \lim_{\xi \rightarrow +\infty} g(\xi) = \mu_i^+, & \lim_{\xi \rightarrow -\infty} g(\xi) = \mu_i^-, \end{cases} \quad (4)$$

where μ^-, μ^+, ζ are constants with $\mu^- < \mu^+$. Typical configurations of the activation function $g(x)$ and its derivative are depicted in Figs. 1 and 2.

Notably, the stationary equation of system (3) is as follows,

$$H_i(x) := -c_i x_i + \sum_{j=1}^n (a_{ij} + b_{ij}) g(x_j) + I_i = 0, \quad i = 1, 2, \dots, n. \quad (5)$$

Define

$$\begin{aligned} h_i^+(\xi) &:= -c_i \xi + (a_{ii} + b_{ii}) g(\xi) + k_i^+, \\ h_i^-(\xi) &:= -c_i \xi + (a_{ii} + b_{ii}) g(\xi) + k_i^-, \\ h_i(\xi) &:= -c_i \xi + (a_{ii} + b_{ii}) g(\xi) + I_i, \end{aligned} \quad i = 1, 2, \dots, n. \quad (6)$$

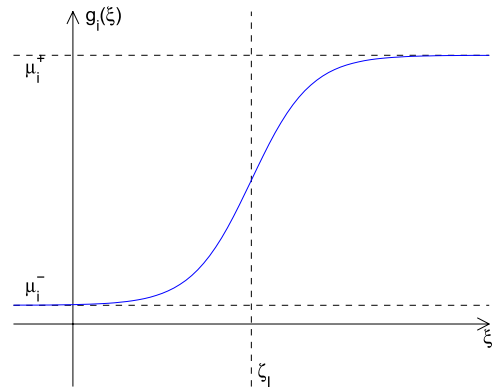


Fig. 1. Configuration of $g_i(\xi)$.

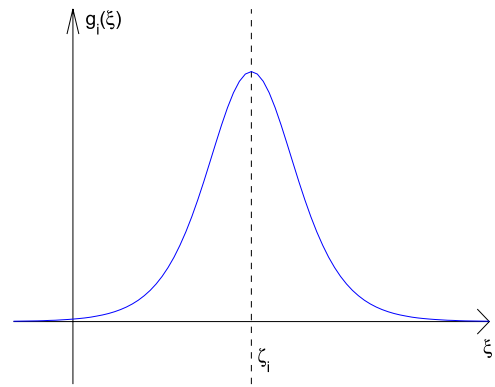


Fig. 2. Configuration of $g_i(\xi)$.

where $k_i^+ = \sum_{j=1, j \neq i}^n (|a_{ij}| + |b_{ij}|) \mu_j + I_i$, $k_i^- = -\sum_{j=1, j \neq i}^n (|a_{ij}| + |b_{ij}|) \mu_j + I_i$ and $\mu_i := \max\{|\mu_i^+|, |\mu_i^-|\}$.

The existence of multiple equilibria are guaranteed by conditions (H₁), (H₂) proposed in Cheng et al. (2007), as follows,

$$(H_1) : \quad 0 < \frac{c_i}{a_{ii} + b_{ii}} < \dot{g}_i(\zeta_i), \quad i = 1, \dots, n;$$

$$(H_2) : \quad h_i^+(p_i) < 0, \quad h_i^-(q_i) > 0, \quad i = 1, \dots, n.$$

According to Proposition 2.1 in Cheng et al. (2007), under condition (H₁), there exist two points p_i and q_i with $p_i < \zeta_i < q_i$, such that $\dot{h}_i(p_i) = \dot{h}_i(q_i) = 0$, $i = 1, \dots, n$. And, from the conditions (H₁), (H₂), there exist points $l_i^+ < m_i^+ < r_i^+$ such that $h_i^+(l_i^+) = h_i^+(m_i^+) = h_i^+(r_i^+) = 0$ as well as points $l_i^- < m_i^- < r_i^-$ such that $h_i^-(l_i^-) = h_i^-(m_i^-) = h_i^-(r_i^-) = 0$. The configuration is depicted in Figs. 3 and 4. Hence, there exists 3^n subset in $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$, denoted by,

$$\begin{aligned} \Lambda^\alpha &= \{\phi = (\phi_1, \dots, \phi_n) | l_i^- < \phi_i(\theta) < l_i^+ \text{ if } \alpha_i = \text{"l"}; \\ & \quad m_i^+ < \phi_i(\theta) < m_i^- \text{ if } \alpha_i = \text{"m"}; r_i^- < \phi_i(\theta) < r_i^+ \text{ if } \alpha_i = \text{"r"}\} \end{aligned} \quad (7)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i = \text{"l"}, \text{"m"}, \text{"r"}$. And, we can have the existence of the equilibria as follows.

Lemma 1 (Theorem 2.2 in Cheng et al. (2007)). *Under the conditions (H₁), (H₂), there exist at least 3^n equilibria for (3), and each of them lies in one of the 3^n regions Λ^α .*

3. Delay-dependent multistability

Assume Λ^α is a subset of $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ defined in (7), where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i = \text{"l"}, \text{"m"}, \text{"r"}$, and $x^* = (x_1^*, \dots, x_n^*)$

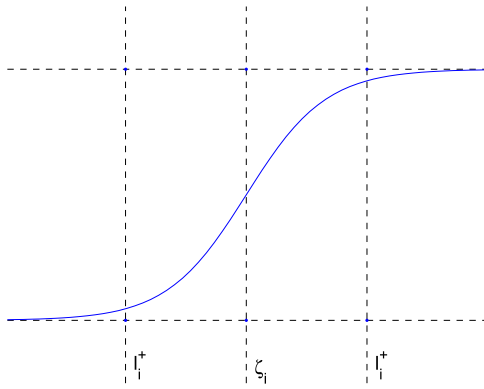


Fig. 3. Configuration of g_i .

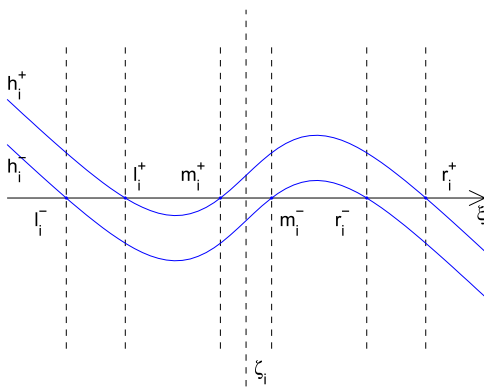


Fig. 4. Configuration of h_i^+ and h_i^- .

is an equilibrium in Λ^α . In this section, we consider the delay-dependent stability of $x^* \in \Lambda^\alpha$.

Consider the stability of x^* in its neighborhood Λ , defined as follows,

$$\Lambda = \{\phi = (\phi_1, \dots, \phi_n) \mid \phi_i \in \mathcal{C}([-\tau, 0], \mathbb{R}) \text{ and } x_i^- \leq \phi_i(\theta) \leq x_i^+, \forall \theta \in [-\tau, 0]\} \quad (8)$$

where $x_i^* \in [x_i^-, x_i^+]$. Hence, there exist two constants $\sigma_i^+ > \sigma_i^- > 0$, such that if $\xi \in [x_i^-, x_i^+]$, then $\dot{g}_i(\xi) \in [\sigma_i^-, \sigma_i^+]$, and $\dot{g}_i(x_i^*) \in [\sigma_i^-, \sigma_i^+]$.

Let $y_i(t) = x_i(t) - x_i^*$, then

$$\dot{y}_i(t) = -c_i y_i(t) + \sum_{j=1}^n a_{ij} f_j(y_j(t)) + \sum_{j=1}^n b_{ij} f_j(y_j(t - \tau)),$$

$$i = 1, 2, \dots, n.$$

which can be rewritten in vector forms as follows:

$$\dot{y}(t) = -Cy(t) + Af(y(t)) + Bf(y(t - \tau)), \quad (9)$$

where $y = (y_1, \dots, y_n)^T$, $f(y(t)) = (f_1(y_1(t)), \dots, f_n(y_n(t)))^T$ and $f_i(y_i(t)) = g_i(x_i(t)) - g_i(x_i^*)$, where

$$\frac{f_i(y_i)}{y_i} = \frac{g_i(y_i + x_i^*) - g_i(x_i^*)}{y_i + x_i^* - y_i} = \dot{g}_i(\xi)$$

and $f_i(0) = 0$. Hence, if $x_i \in [x_i^-, x_i^+]$, then $\xi \in [x_i^-, x_i^+]$ and

$$\sigma_i^- \leq \frac{f_i(y_i(t))}{y_i(t)} \leq \sigma_i^+. \quad (10)$$

Denote

$$\begin{aligned} \Sigma_1 &= \text{diag}(\sigma_1^+ \sigma_1^-, \dots, \sigma_n^+ \sigma_n^-), \\ \Sigma_2 &= \text{diag}\left(\frac{\sigma_1^+ + \sigma_1^-}{2}, \dots, \frac{\sigma_n^+ + \sigma_n^-}{2}\right). \end{aligned} \quad (11)$$

To prove our main theorem, we also need the following lemmas.

Lemma 2. For any diagonal matrices $U = \text{diag}(u_1, \dots, u_n) > 0$, $V = \text{diag}(v_1, \dots, v_n) > 0$, if (10) holds, then

$$\begin{aligned} & \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix}^T \begin{bmatrix} -U\Sigma_1 & U\Sigma_2 \\ U\Sigma_2 & -U \end{bmatrix} \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix} \\ & + \begin{bmatrix} y(t - \tau) \\ f(y(t - \tau)) \end{bmatrix}^T \begin{bmatrix} -V\Sigma_1 & V\Sigma_2 \\ V\Sigma_2 & -V \end{bmatrix} \begin{bmatrix} y(t - \tau) \\ f(y(t - \tau)) \end{bmatrix} \geq 0. \end{aligned} \quad (12)$$

The proof can be seen in Liu, Wang, Serrano, and Liu (2007).

Lemma 3. For real symmetric matrices $K > 0$, $M_i (i = 1, 2, 3, 4)$ with appropriate dimensions, then

$$-\int_{t-\tau}^t \dot{y}^T(s)K\dot{y}(s)ds \leq \xi^T(t)[- \tau M^T K^{-1} M + M^T \hat{J} + \hat{J}^T M] \xi(t), \quad (13)$$

where

$$\begin{aligned} \xi(t) &= [y^T(t), y^T(t - \tau), f^T(y(t)), f^T(y(t - \tau))]^T, \\ M &= [M_1, M_2, M_3, M_4], \\ \hat{J} &= [I, -I, 0, 0]^T. \end{aligned}$$

Proof. Note the fact that

$$\begin{aligned} & \begin{bmatrix} I & -M^T K^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} M^T K^{-1} M & M^T \\ M & K \end{bmatrix} \begin{bmatrix} I & -M^T K^{-1} \\ 0 & I \end{bmatrix}^T \\ & = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \geq 0, \end{aligned}$$

then one has

$$\begin{bmatrix} M^T K^{-1} M & M^T \\ M & K \end{bmatrix} \geq 0.$$

It follows that

$$\begin{aligned} 0 & \leq \int_{t-\tau}^t \begin{bmatrix} \xi(t) \\ \dot{y}(s) \end{bmatrix}^T \begin{bmatrix} M^T K^{-1} M & M^T \\ M & K \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{y}(s) \end{bmatrix} ds \\ & \leq \tau \xi^T(t) M^T K^{-1} M \xi(t) + \xi^T(t) M^T \int_{t-\tau}^t \dot{y}(s) ds \\ & \quad + \int_{t-\tau}^t \dot{y}^T(s) ds M \xi(t) + \int_{t-\tau}^t \dot{y}(s) K \dot{y}(s) ds \\ & \leq \tau \xi^T(t) M^T K^{-1} M \xi(t) + \xi^T(t) [\hat{J} M + (\hat{J} M)^T] \xi(t) \\ & \quad + \int_{t-\tau}^t \dot{y}(s) K \dot{y}(s) ds. \end{aligned}$$

Obviously, (13) holds. This completes the proof.

Lemma 4 (Schur Complement, Boyd (1994)). Given constant symmetric matrices $\Omega_1, \Omega_2, \Omega_3$, where $\Omega_1 = \Omega_1^T$ and $\Omega_2 = \Omega_2^T > 0$, then

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0.$$

Theorem 1. For given τ , the equilibrium x^* in Λ is asymptotically stable, if Λ is positively invariant for (3) and there exist three symmetric matrices $P > 0, Q > 0, K > 0$, two diagonal matrices $U > 0, V > 0$, and $M_i (i = 1, \dots, 4)$ with appropriate dimensions such that the LMI (14) holds

$$\begin{bmatrix} \Omega + \hat{J}M + (\hat{J}M)^T & \tau M^T \\ \tau M & -\tau K \end{bmatrix} < 0, \tag{14}$$

where

$$\Omega = \begin{bmatrix} \Omega_{11} & 0 & PA + U\Sigma_2 - \tau C^T KA & PB - \tau C^T KB \\ * & -Q - V\Sigma_1 & 0 & V\Sigma_2 \\ * & * & -U + \tau A^T KA & \tau A^T KB \\ * & * & * & -V + \tau B^T KB \end{bmatrix},$$

$$\Omega_{11} = -(C^T P + PC) + Q - U\Sigma_1 + \tau C^T KC,$$

$M = [M_1, M_2, M_3, M_4]$, $\hat{J} = [I, -I, 0, 0]^T$ and Σ_1, Σ_2 are defined in (11). Here, * denotes the transpose of the corresponding upper diagonal elements of the matrix.

Proof. Choose a Lyapunov–Krasovskii functional candidate as

$$V(t) = y^T(t)Py(t) + \int_{t-\tau}^t y^T(s)Qy(s)ds + \int_{-\tau}^0 \int_{t+\theta}^t \dot{y}^T(s)K\dot{y}(s)dsd\theta, \tag{15}$$

where $P = P^T > 0, Q = Q^T > 0, K = K^T > 0$. Employing Lemmas 2 and 3, calculating the time-derivative of $V(t)$ along the trajectories,

$$\begin{aligned} \dot{V}(t) &= 2y^T(t)P\dot{y}(t) + y^T(t)Qy(t) - y^T(t-\tau)Qy(t-\tau) \\ &\quad + \int_{-\tau}^0 \dot{y}^T(t)K\dot{y}(t)d\theta - \int_{-\tau}^0 \dot{y}^T(t+\theta)K\dot{y}(t+\theta)d\theta \\ &= 2y^T(t)P[-Cy(t) + Af(y) + Bf(y(t-\tau))] + y^T(t)Qy(t) \\ &\quad - y^T(t-\tau)Qy(t-\tau) + \tau[-Cy(t) + Af(y) + Bf(y(t-\tau))]^T \\ &\quad \times K[-Cy(t) + Af(y) + Bf(y(t-\tau))] - \int_{t-\tau}^t \dot{y}^T(s)K\dot{y}(s)ds \\ &\leq 2y^T(t)P[-Cy(t) + Af(y) + Bf(y(t-\tau))] + y^T(t)Qy(t) \\ &\quad - y^T(t-\tau)Qy(t-\tau) + \tau[-Cy(t) + Af(y) + Bf(y(t-\tau))]^T \\ &\quad \times K[-Cy(t) + Af(y) + Bf(y(t-\tau))] \\ &\quad + \xi^T(t)[- \tau M^T K^{-1} M + \hat{J}M + (\hat{J}M)^T] \xi(t) \\ &\quad + \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix}^T \begin{bmatrix} -U\Sigma_1 & U\Sigma_2 \\ U\Sigma_2 & -U \end{bmatrix} \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix} \\ &\quad + \begin{bmatrix} y(t-\tau) \\ f(y(t-\tau)) \end{bmatrix}^T \begin{bmatrix} -V\Sigma_1 & V\Sigma_2 \\ V\Sigma_2 & -V \end{bmatrix} \begin{bmatrix} y(t-\tau) \\ f(y(t-\tau)) \end{bmatrix} \\ &= \xi^T(t)[\Omega + \hat{J}M + (\hat{J}M)^T - \tau M^T K^{-1} M] \xi(t). \end{aligned}$$

Employing Schur complement in Lemma 4, $\Omega + \hat{J}M + (\hat{J}M)^T - \tau M^T K^{-1} M < 0$ is guaranteed by LMI (14). If $\Omega + \hat{J}M + (\hat{J}M)^T - \tau M^T K^{-1} M < 0$, it yields $\dot{V}(t) < 0$ when $x_t \in \Lambda$, which implies x^* in the positive invariant set Λ is locally asymptotically stable and thus the proof is completed.

If we choose Lyapunov–Krasovskii functional (15) with $K = 0$, then the obtained stability criterion is delay-independent. The proof is similar to Theorem 1.

Theorem 2. The equilibrium x^* in Λ is asymptotically stable, if Λ is positively invariant for (3) and there exist three symmetric matrices $P > 0, Q > 0$, two diagonal matrices $U > 0, V > 0$ with appropriate dimensions such that the LMI (16) holds

$$\Omega < 0 \tag{16}$$

$$= \begin{bmatrix} -(C^T P + PC) + Q - U\Sigma_1 & 0 & PA + U\Sigma_2 & PB \\ * & -Q - V\Sigma_1 & 0 & V\Sigma_2 \\ * & * & -U & 0 \\ * & * & * & -V \end{bmatrix}$$

where Σ_1, Σ_2 are defined in (11).

Remark 1. In the multistability criteria of Theorems 1 and 2, the choice of the positive invariant set Λ is much vital and flexible. On one hand, it is related to the stability discrimination of the equilibria. A smaller Λ can make the corresponding matrix Σ_1, Σ_2 more precise and the stability of the equilibria x^* in Λ will be determined more accurately. On the other hand, if the equilibria x^* is stable for the criterion above, then the the positive invariant set Λ is the attraction domain of x^* . If the inequality (14) or (16) holds on a larger Λ , then the attraction domain of x^* will be larger.

Remark 2. While considering stability for a large number of equilibria, we can take the different Σ_1, Σ_2 for each equilibria and solve every LMI to determine the stability for all equilibria. The computational complexity would be very high. For instance, if there exist 2^n equilibria, then we need to verify 2^n LMIs. In another way, we can set common Σ_1, Σ_2 for all equilibria in their neighborhood and examine just one LMI to verify whether all equilibria are stable. While the common Σ_1, Σ_2 would be more rough in general.

In the following, we will discuss the condition for Λ to be the positive invariant set.

$$(H_3) : \begin{cases} -c_i x_i^+ + \sum_{j=1}^n (a_{ij}^+ + b_{ij}^+) g_j(x_j^+) \\ \quad + \sum_{j=1}^n (a_{ij}^- + b_{ij}^-) g_j(x_j^-) + I_i < 0, \\ -c_i x_i^- + \sum_{j=1}^n (a_{ij}^+ + b_{ij}^+) g_j(x_j^-) \\ \quad + \sum_{j=1}^n (a_{ij}^- + b_{ij}^-) g_j(x_j^+) + I_i > 0, \end{cases} \quad i = 1, 2, \dots, n.$$

Here, $a_{ij}^+ = \max\{0, a_{ij}\}, a_{ij}^- = \min\{0, a_{ij}\}, b_{ij}^+ = \max\{0, b_{ij}\}, b_{ij}^- = \min\{0, b_{ij}\}$.

Lemma 5. Under the condition H_3 , Λ is positively invariant for (3)

Proof. If Λ is not positively invariant, then there must exist a solution $x(t)$ with its initial value $\phi = (\phi_1, \dots, \phi_n) \in \Lambda$, which leaves the region Λ first at time $t_0 > 0$. Without losing generality, assume x_i leaves $[x_i^-, x_i^+]$ first. Then $x_i(t_0) = x_i^+, \dot{x}_i(t_0) > 0$, or $x_i(t_0) = x_i^-, \dot{x}_i(t_0) < 0$.

Considering $x_i(t_0) = x_i^+$, from condition (H_3) we have

$$\begin{aligned} \dot{x}_i(t_0) &= -c_i x_i(t_0) + \sum_{j=1}^n a_{ij} g_j(x_j(t_0)) + \sum_{j=1}^n b_{ij} g_j(x_j(t_0 - \tau)) + I_i \\ &= -c_i x_i^+ + \sum_{j=1}^n (a_{ij}^+ + a_{ij}^-) g_j(x_j(t_0)) \\ &\quad + \sum_{j=1}^n (b_{ij}^+ + b_{ij}^-) g_j(x_j(t_0 - \tau)) + I_i \\ &\leq -c_i x_i^+ + \sum_{j=1}^n (a_{ij}^+ + b_{ij}^+) g_j(x_j^+) + \sum_{j=1}^n (a_{ij}^- + b_{ij}^-) g_j(x_j^-) + I_i \\ &< 0, \end{aligned}$$

It is inconsistent with $\dot{x}_i(t_0) > 0$. We can also have the same conclusion for the condition $x_i(t_0) = x_i^-$. Hence, $x(t)$ cannot leave Λ , and Λ is positively invariant for (3). This completes the proof.

Corollary 1. For given τ , the equilibrium x^* in Λ is asymptotically stable, if condition (H₃) holds and there exist three symmetric matrices $P > 0, Q > 0, K > 0$, two diagonal matrices $U > 0, V > 0$, and M_i ($i = 1, \dots, 4$) with appropriate dimensions such that the LMI (14) holds.

Corollary 2. The equilibrium x^* in Λ is asymptotically stable, if condition (H₃) holds and there exist three symmetric matrices $P > 0, Q > 0$, two diagonal matrices $U > 0, V > 0$ with appropriate dimensions such that the LMI (16) holds.

Furthermore, under the conditions (H₁), (H₂), the regions Λ^α can be divided into two classes.

$$\Lambda_1 = \{\Lambda^\alpha \mid \exists i \leq n, \text{ s.t. } \alpha_i = "m"\},$$

$$\Lambda_2 = \{\Lambda^\alpha \mid \alpha_i = "l" \text{ or } "r" \forall i \leq n\}.$$

Λ_2 is composed of 2^n regions and Λ_1 is composed of $3^n - 2^n$ regions. If the equilibrium point $x^* \in \Lambda^\alpha$ and $\Lambda^\alpha \in \Lambda_1$, then x^* is usually unstable under the conditions (H₁), (H₂). Hence, we only consider the equilibrium points in Λ^α , where $\Lambda^\alpha \in \Lambda_2$.

Lemma 6. Under the conditions (H₁), (H₂), if $b_{ii} > 0$ for any $i = \dots, n$, then every $\Lambda^\alpha \in \Lambda_2$ is positively invariant for (3).

Proof. If Λ^α is not positively invariant, then there must exist a solution $x(t)$ with its initial value $\phi = (\phi_1, \dots, \phi_n) \in \Lambda^\alpha$, which leaves the region Λ^α first at time $t_0 > 0$. Without losing generality, assume x_i leaves $[x_i^-, x_i^+]$ first and $\alpha_i = "l"$. Then $x_i(t_0) = l_i^+, \dot{x}_i(t_0) > 0$, or $x_i(t_0) = l_i^-, \dot{x}_i(t_0) < 0$.

Considering $x_i(t_0) = l_i^+$, from condition (H₂) we have

$$\begin{aligned} \dot{x}_i(t_0) &= -c_i x_i(t_0) + \sum_{j=1}^n a_{ij} g_j(x_j(t_0)) + \sum_{j=1}^n b_{ij} g_j(x_j(t_0 - \tau)) + I_i \\ &\leq -c_i l_i^+ + (a_{ii} + b_{ii}) g_i(l_i^+) + \sum_{j=1, j \neq i}^n (|a_{ij}| + |b_{ij}|) \mu_j + I_i \\ &= h_i^+(l_i^+) \\ &= 0. \end{aligned}$$

It is inconsistent with $\dot{x}_i(t_0) > 0$. We can also have the same conclusion for the condition $x_i(t_0) = l_i^-$. Hence, $x(t)$ cannot leave Λ^α , and Λ^α is positively invariant for (3). This completes the proof.

Corollary 3. Under the conditions (H₁), (H₂), take $\sigma_i^- = \min\{\dot{g}(l_i^-), \dot{g}(r_i^+)\}$, $\sigma_i^+ = \max\{\dot{g}(l_i^+), \dot{g}(r_i^-)\}$. For given $\tau \geq 0$, there are 2^n local asymptotically stable equilibria, if $b_{ii} > 0$ for any $i = \dots, n$ and there exist three symmetric matrices $P > 0, Q > 0, K > 0$, two diagonal matrices $U > 0, V > 0$, and M_i ($i = 1, \dots, 4$) with appropriate dimensions such that the LMI (14) holds.

Proof. For any $\phi \in \Lambda^\alpha$, where $\Lambda^\alpha \in \Lambda_2$, we have

$$\min\{\dot{g}(l_i^-), \dot{g}(r_i^+)\} < \phi_i(\theta) < \max\{\dot{g}(l_i^+), \dot{g}(r_i^-)\}, \quad \forall \theta \in [-\tau, 0].$$

Hence, Corollary 3 can be derived directly from Theorem 1 and Lemma 6.

Corollary 4. Under the conditions (H₁), (H₂), take $\sigma_i^- = \min\{\dot{g}(l_i^-), \dot{g}(r_i^+)\}$, $\sigma_i^+ = \max\{\dot{g}(l_i^+), \dot{g}(r_i^-)\}$. For given $\tau \geq 0$, there are 2^n local asymptotically stable equilibria, if $b_{ii} > 0$ for any $i = \dots, n$ and there exist three symmetric matrices $P > 0, Q > 0, K > 0$, two diagonal matrices $U > 0, V > 0$ with appropriate dimensions such that the LMI (16) holds.

Remark 3. In Theorem 3.2 (Cheng et al., 2007), the delay-independent multistability condition is obtained under the assumptions (H₁), (H₂), $b_{ii} > 0$ and

$$c_i > \sum_{j=1}^n \eta_j (|a_{ij}| + |b_{ij}|), \quad \text{for } i = 1, 2, \dots, n, \tag{17}$$

where $\max\{g'_j(l_j^+), g'_j(r_j^-)\} < \eta_j < \min\{g'_j(p_j), g'_j(q_j)\}$, $j = 1, \dots, n$. It is obvious that the signs of the weight connections and the delayed weight connections are neglected in the conditions above, that is to say, the differences between the neuronal excitatory and the inhibitory effects have been neglected. While, by using Lyapunov–Krasovskii stability theorem and LMI method, our criteria avoid this problem.

4. Two examples

In this section, two examples are presented to illustrate both delay-independent and delay-dependent multistability results.

Example 1. Consider Example 6.4 in Cheng et al. (2007) as follows

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + 7g_1(x_1(t)) + 0.5g_2(x_2(t)) \\ \quad - 4g_1(x_1(t - \tau_{11})) + 0.5g_2(x_2(t - \tau_{12})), \\ \dot{x}_2(t) = -x_2(t) + 0.5g_1(x_1(t)) + 7g_2(x_2(t)) \\ \quad + 0.5g_1(x_1(t - \tau_{21})) - 4g_2(x_2(t - \tau_{22})), \end{cases} \tag{18}$$

where $g_1(x) = g_2(x) = \tanh(x)$. Here, we assume $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = \tau$. With the same computation in Cheng et al. (2007), conditions (H₁), (H₂) are satisfied, and $l_1^+ = l_2^+ = -1.8573$, $m_1^+ = m_2^+ = -0.5903$, $r_1^+ = r_2^+ = 3.9980$; $l_1^- = l_2^- = -3.9980$, $m_1^- = m_2^- = 0.5903$, $r_1^- = r_2^- = 1.8573$.

As the analysis and simulations in Example 6.4 Cheng et al. (2007), the equilibrium in $\Omega^{(l,l)}, \Omega^{(r,l)}, \Omega^{(l,r)}, \Omega^{(r,r)}$ is stable with $\tau_{11} < 0.08475$, $\tau_{22} < 0.08475$.

In fact, for each equilibria, we can make the matrix Σ_1, Σ_2 more precise. For the equilibrium (3.9973, 3.9973), choose the $[x_1^-, x_1^+] \times [x_2^-, x_2^+]$ in (8) as $[3.9, 4.1] \times [3.9, 4.1]$ and $\sigma_1^+ = \sigma_2^+ = 1.6376 \times 10^{-3}$, $\sigma_1^- = \sigma_2^- = 1.0980 \times 10^{-3}$,

$$\begin{aligned} \Sigma_1 &= \begin{pmatrix} 1.798 \times 10^{-4} & 0 \\ 0 & 1.798 \times 10^{-4} \end{pmatrix}, \\ \Sigma_2 &= \begin{pmatrix} 1.367 \times 10^{-3} & 0 \\ 0 & 1.367 \times 10^{-3} \end{pmatrix}. \end{aligned}$$

For the equilibrium (1.9150, -1.9150), choose the $[x_1^-, x_1^+] \times [x_2^-, x_2^+]$ in (8) as $[1.91, 1.92] \times [1.91, 1.92]$ and $\sigma_1^+ = \sigma_2^+ = 8.2394 \times 10^{-2}$, $\sigma_1^- = \sigma_2^- = 8.3987 \times 10^{-2}$,

$$\begin{aligned} \Sigma_1 &= \begin{pmatrix} 6.920 \times 10^{-3} & 0 \\ 0 & 6.920 \times 10^{-3} \end{pmatrix}, \\ \Sigma_2 &= \begin{pmatrix} 8.319 \times 10^{-2} & 0 \\ 0 & 8.319 \times 10^{-2} \end{pmatrix}. \end{aligned}$$

And, it can be verified that condition (H₃) holds for the two equilibria. From Corollary 2, they are all asymptotically stable for any $\tau \geq 0$. To the other two equilibria, we can have the same conclusion. Different from the simulation in Cheng et al. (2007), Figs. 5 and 6 depict the dynamics with $\tau = 10$, and the attraction basins for (1.9150, -1.9150) and (-1.9150, 1.9150) are really small.

Example 2. Consider the neural network as follows

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + 4g_1(x_1(t)) - 2g_2(x_2(t)) \\ \quad + 3g_1(x_1(t - \tau)) + 2.8g_2(x_2(t - \tau)), \\ \dot{x}_2(t) = -x_2(t) - 2g_1(x_1(t)) + 4g_2(x_2(t)) \\ \quad + 2.8g_1(x_1(t - \tau)) + 3g_2(x_2(t - \tau)), \end{cases} \tag{19}$$

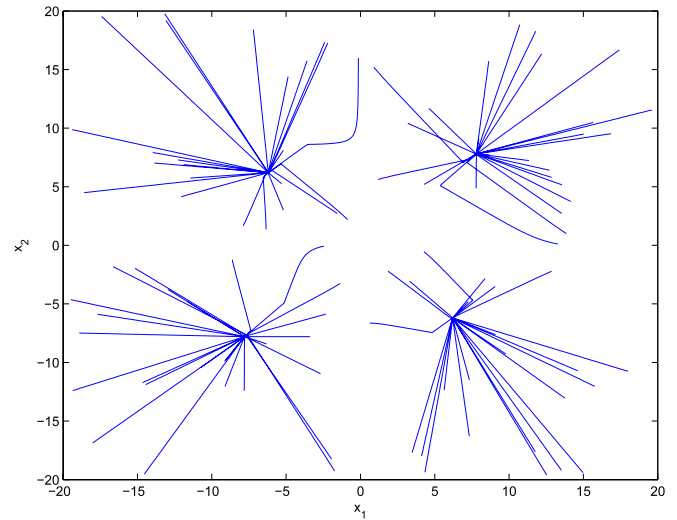


Fig. 7. Phase plot of (x_1, x_2) in Example 2 with $\tau = 10$.

Fig. 5. Phase plot of (x_1, x_2) in Example 1 with $\tau = 10$. The subfigures plot the dynamic behaviors near the equilibria $(1.9150, -1.9150)$ and $(-1.9150, 1.9150)$. The trajectories in the same color converge to the same stable equilibrium point.

Fig. 6. Response of x_1, x_2 in Example 1 with $\tau = 10$.

Fig. 8. Time response of x_1, x_2 in Example 2 with $\tau = 10$.

where $g_1(x) = g_2(x) = \tanh(x)$ and $\tau = 10$. Direct computation gives

$$h_1^+(\xi) = h_2^+(\xi) = -\xi + 7 \tanh(\xi) + 4.8,$$

$$h_1^-(\xi) = h_2^-(\xi) = -\xi + 7 \tanh(\xi) - 4.8.$$

The parameters satisfy the criterion in Corollary 3:

$$\text{Condition}(H_1) : 0 < \frac{c_1}{a_{11} + b_{11}} = \frac{1}{7} < 1,$$

$$0 < \frac{c_2}{a_{22} + b_{22}} = \frac{1}{7} < 1;$$

$$\text{Condition}(H_2) : p_1 = p_2 = -1.6283, \quad q_1 = q_2 = 1.6283,$$

$$h_1^+(p_1) = h_2^+(p_2) = -0.0524 < 0,$$

$$h_1^-(q_1) = h_2^-(q_2) = 0.0524 > 0.$$

And, $l_1^+ = l_2^+ = -1.8838, m_1^+ = m_2^+ = -1.4050, r_1^+ = r_2^+ = 11.8000; l_1^- = l_2^- = -11.8000, m_1^- = m_2^- = 1.4050, r_1^- = r_2^- = 1.8838$. Set

$$\Sigma_1 = \begin{pmatrix} 1.989 \times 10^{-9} & 0 \\ 0 & 1.989 \times 10^{-9} \end{pmatrix},$$

$$\Sigma_2 = \begin{pmatrix} 0.0442 & 0 \\ 0 & 0.0442 \end{pmatrix},$$

thus we can easily verify that the LMI (14) is satisfied. From Corollary 3, there exists 2^n stable equilibria. The parameters herein do not satisfy the criterion (17) for theory in Cheng et al. (2007):

$$\begin{aligned} & \eta_1(|a_{11}| + |b_{11}|) + \eta_2(|a_{12}| + |b_{12}|) \\ & > \max\{g_1'(l_1^+), g_1'(r_1^-)\}(|a_{11}| + |b_{11}|) \\ & \quad + \max\{g_2'(l_2^+), g_2'(r_2^-)\}(|a_{12}| + |b_{12}|) \\ & = 1.0420 > 1 \\ & = c_1, \end{aligned}$$

$$\begin{aligned} & \eta_1(|a_{21}| + |b_{21}|) + \eta_2(|a_{22}| + |b_{22}|) \\ & > \max\{g_1'(l_1^+), g_1'(r_1^-)\}(|a_{21}| + |b_{21}|) \\ & \quad + \max\{g_2'(l_2^+), g_2'(r_2^-)\}(|a_{22}| + |b_{22}|) \\ & = 1.0420 > 1 \\ & = c_2, \end{aligned}$$

which demonstrate the assertion in Remark 3. The dynamics of (19) are illustrated in Figs. 7 and 8.

5. Coexistence of equilibria and limited cycles

In this section, we consider a special case to show the coexistence of equilibria and limited cycles.

