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Delay-dependent multistability in recurrent neural networks[☆]

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ABSTRACT

In this article, we focus on the delay-dependent multistability in recurrent neural networks. By constructing Lyapunov functional and using matrix inequality techniques, a novel delay-dependent multistability criterion is derived. The obtained results are more flexible and less conservative than previously known criteria. Two examples are given to show the effectiveness of the obtained criteria. Furthermore, some interesting delay-dependent dynamic behaviors have been showed in a special case, for example, we find that there is the coexistence of stable equilibria and stable limit cycles in the single neuron. Also, when the neurons are coupled, then the stable patterns are more complex.

1. Introduction

In recent years, neural networks have attracted more and more attention of researchers. Ranging from signal processing, pattern recognition, programming problems and static image processing, neural networks have witnessed a large amount of successful applications in many fields (see Cichocki, 2002; Cochocki & Unbehauen, 1993; Forti, Nistri, & Quincampoix, 2004; Forti & Tesi, 1995; Karhunen, Hyvarinen, Vigario, Hurri, & Oja, 1997; Xia, Leung, & Bosse, 2002). And, these applications depend heavily on the network's dynamics. As practical applications of neural networks, multistability is a necessary feature for associative memory storage and pattern recognition. Multistability describes the coexistence of multiple stable patterns (Chua, 1998; Foss, Longtin, Mensour, & Milton, 1996; Hopfield, 1984; Morita, 1993), including stable equilibria and stable limit cycles. In Cohen (1992), two distinct but related constructive methods are provided for constructing systems of ordinary differential equations with arbitrary numbers of stable patterns.

Multistability in the delayed neural networks:

$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j(x_j(t-\tau_{ij})) + J_i, \quad i = 1, 2, \ldots, n \quad (1)$$

is discussed by Cheng, Lin, and Shih (2006). It is found that an n-neuron cellular neural networks can have up to 2^n locally stable equilibria. And, Cheng, Lin, and Shih (2007) has studied a general delayed neural networks:

$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} \beta_{ij} g_j(x_j(t-\tau_{ij})) + J_i,$$
$$i = 1, 2, \ldots, n. \quad (2)$$

In addition, the multistability of cellular neural networks with and without delays is investigated by Zeng, Wang, and Liao (2004) and Zeng, Huang, and Wang (2005). Furthermore, Zeng and Wang (2006) and Cheng et al. (2007) investigated the conditions for the existence of multiple stable periodic orbits evoked by periodic external inputs, in which all the stable patterns are limit cycles with the same periodic time. And, if orbits converge to the same periodic orbits, then they will be synchronized.

Since time delays are often encountered due to measurement and computational delays, which may result in oscillation and instability, the stability and multistability analyses of delayed neural networks have received considerable attention (e.g., see Cao, Ho, & Huang, 2007; Cao & Li, 2005; He & Wu, 2006; Li & Chen, 2007; Lou & Cui, 2006; Singh, 2006; Wang, Shu, Liu, Ho, & Liu, 2006; Xu & Lam, 2006). However, most investigations on multistability have focused on the delay-independent stability analysis. In general, the delay-dependent stability criteria are less conservative than delay-independent ones. Though Cheng et al. (2007) obtained a delay-dependent multistability criterion using their theory of quasiconvergence, there was a strong possibility that their criterion may be more conservative since they result from the strongly order preserving of the semiflow generated by the solution of neural networks (2). To overcome this conservatism, we shall derive

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a new delay-dependent multistability criterion for the neural networks by utilizing Linear Matrix Inequality (LMI) convex optimization approach.

Recently, the LMI-based techniques have been successfully used to tackle various stability problems for neural networks with or without delay (see Cao et al., 2007; Cao & Li, 2005; Lou & Cui, 2006; Wang et al., 2006; Xu & Lam, 2006). The main advantage of the LMI-based approaches is that the LMI stability conditions can be solved numerically using the effective interior-point algorithm. And, the delay-dependent stability is considered for neural networks based on LMI approach in Xu and Lam (2006) and Wang et al. (2006). Beside the stability problems, the LMI approaches (e.g. Lu & Chen, 2004; Yu & Cao, 2007) have also been used successfully to synchronize and estimate the state of the respective neural networks (see He, Wang, Wu, & Lin, 2006; Wang, Ho, & Liu, 2005).

Furthermore, to learn how delay effect on the multistability of the neural network, a special case will be investigated. Associating with the multistability criteria derived in this article and numerical simulations, we shall explore an interesting phenomenon that there coexist two stable equilibrium points and one stable limit cycle in a single neuron. It is different from the Hopf bifurcation (see Hassard, Kazarinoff, & Wan, 1981; Song, Han, & Wei, 2005; Zhu & Huang, 2007) and the other coexisting phenomenon for stable patterns (e.g. Campbell, Ncube, & Wu, 2006). Consider the neural network coupled by two neurons with small connection strength. Besides the compound stable patterns from the existing stable patterns of the single neuron, some new stable patterns emerge. There exist several types of stable patterns, which contain both stable equilibrium points and stable limit cycles with different periodic times. Different from the stable periodic orbits evoked by periodic external inputs, the orbits, which converge to the same stable pattern, can be asynchronous.

The rest of the article is organized as follows. In Section 2, the existence of multiple equilibria is introduced. In Section 3, the delay-dependent multistability criteria are derived. And, two numerical examples are illustrated in Section 4. In Section 5, we give an example to show the coexistence of different types of stable patterns. Finally, the conclusions are given in Section 6.

2. Multiple equilibria

Consider the neural network with delay as follows,

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} g(x_j(t)) + \sum_{j=1}^{n} b_{ij} g(x_j(t-\tau)) + I_i,\quad i = 1, 2, \ldots, n. \tag{3}$$

And, we assume the activation functions $g(x)$ have the sigmoidal configuration, which satisfies the following properties:

$$g \in \mathcal{C}^2, \begin{cases} \mu_i^- < g(\xi) < \mu_i^+, & \dot{g}(\xi) > 0, \\ (\xi - \zeta_i)\ddot{g}(\xi) < 0 & \text{for all } \xi \in \mathbb{R}, \\ \lim\limits_{\xi \to +\infty} g(\xi) = \mu_i^+, & \lim\limits_{\xi \to -\infty} g(\xi) = \mu_i^-, \end{cases} \tag{4}$$

where μ^-, μ^+, ζ are constants with $\mu^- < \mu^+$. Typical configurations of the activation function $g(x)$ and its derivative are depicted in Figs. 1 and 2.

Notably, the stationary equation of system (3) is as follows,

$$H_i(x) := -c_i x_i + \sum_{j=1}^{n} (a_{ij} + b_{ij}) g(x_j) + I_i = 0,\quad i = 1, 2, \ldots, n. \tag{5}$$

Define

$$\begin{aligned} h_i^+(\xi) &:= -c_i\xi + (a_{ii} + b_{ii})g(\xi) + k_i^+, \\ h_i^-(\xi) &:= -c_i\xi + (a_{ii} + b_{ii})g(\xi) + k_i^-, \quad i = 1, 2, \ldots, n. \\ h_i(\xi) &:= -c_i\xi + (a_{ii} + b_{ii})g(\xi) + I_i, \end{aligned} \tag{6}$$

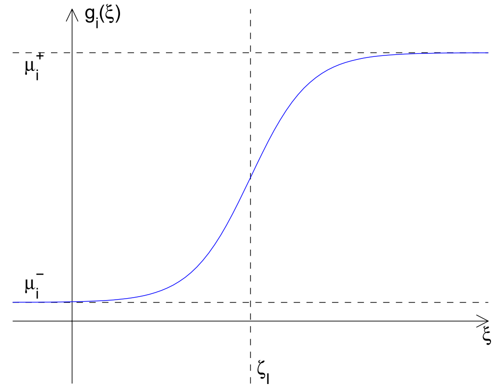

Fig. 1. Configuration of $g_i(\xi)$.

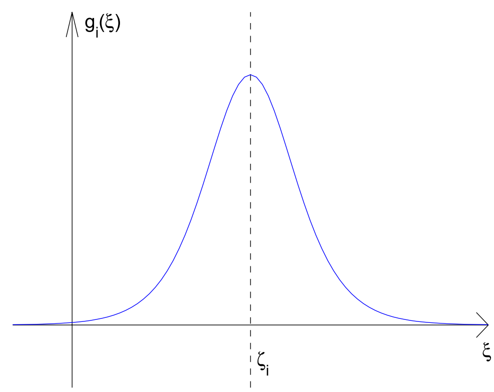

Fig. 2. Configuration of $\dot{g}_i(\xi)$.

where $k_i^+ = \sum_{j=1, j\neq i}^{n} (|a_{ij}| + |b_{ij}|)\mu_j + I_i$, $k_i^- = -\sum_{j=1, j\neq i}^{n} (|a_{ij}| + |b_{ij}|)\mu_j + I_i$ and $\mu_i := \max\{|\mu_i^+|, |\mu_i^-|\}$.

The existence of multiple equilibria are guaranteed by conditions (H_1), (H_2) proposed in Cheng et al. (2007), as follows,

$$(\mathrm{H}_1):\quad 0 < \frac{c_i}{a_{ii} + b_{ii}} < \dot{g}_i(\zeta_i),\quad i = 1, \ldots, n;$$

$$(\mathrm{H}_2):\quad h_i^+(p_i) < 0,\qquad h_i^-(q_i) > 0,\quad i = 1, \ldots, n.$$

According to Proposition 2.1 in Cheng et al. (2007), under condition (H_1), there exist two points p_i and q_i with $p_i < \zeta_i < q_i$, such that $\dot{h}_i(p) = \dot{h}_i(q) = 0$, $i = 1, \ldots, n$. And, from the conditions (H_1), (H_2), there exist points $l_i^+ < m_i^+ < r_i^+$ such that $h_i^+(l_i^+) = h_i^+(m_i^+) = h_i^+(r_i^+) = 0$ as well as points $l_i^- < m_i^- < r_i^-$ such that $h_i^-(l_i^-) = h_i^-(m_i^-) = h_i^-(r_i^-) = 0$. The configuration is depicted in Figs. 3 and 4. Hence, there exists 3^n subset in $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$, denoted by,

$$\begin{aligned} \Lambda^\alpha = \{&\phi = (\phi_1, \ldots, \phi_n) | l_i^- < \phi_i(\theta) < l_i^+ \text{ if } \alpha_i = \text{“}l\text{”}; \\ &m_i^+ < \phi_i(\theta) < m_i^- \text{ if } \alpha_i = \text{“}m\text{”};\ r_i^- < \phi_i(\theta) < r_i^+ \text{ if } \alpha_i = \text{“}r\text{”}\} \end{aligned} \tag{7}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i =$ “l”, “m”, “r”. And, we can have the existence of the equilibria as follows.

Lemma 1 (*Theorem 2.2 in Cheng et al. (2007)*). *Under the conditions* (H_1), (H_2), *there exist at least* 3^n *equilibria for* (3), *and each of them lies in one of the* 3^n *regions* Λ^α.

3. Delay-dependent multistability

Assume Λ^α is a subset of $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ defined in (7), where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i =$ “l”, “m”, “r”, and $x^* = (x_1^*, \ldots, x_n^*)$

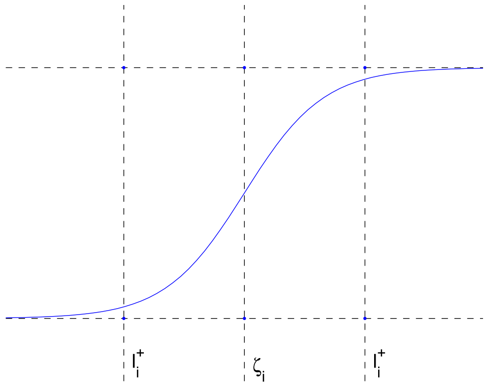

Fig. 3. Configuration of g_i.

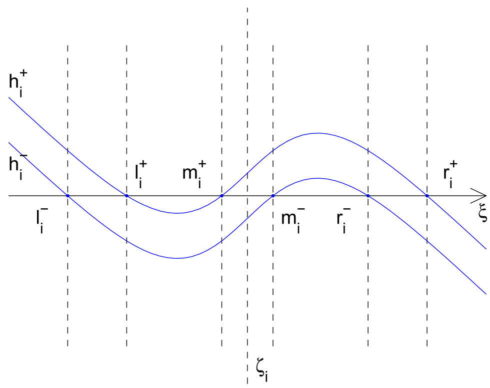

Fig. 4. Configuration of h_i^+ and h_i^-.

is an equilibrium in Λ^α. In this section, we consider the delay-dependent stability of $x^* \in \Lambda^\alpha$.

Consider the stability of x^* in its neighborhood Λ, defined as follows,

$$\Lambda = \{\phi = (\phi_1, \ldots, \phi_n) \mid \phi_i \in \mathcal{C}([-\tau, 0], \mathbb{R}) \text{ and } x_i^- \le \phi_i(\theta) \le x_i^+, \forall\, \theta \in [-\tau, 0]\} \tag{8}$$

where $x_i^* \in [x_i^-, x_i^+]$. Hence, there exist two constants $\sigma_i^+ > \sigma_i^- > 0$, such that if $\xi \in [x_i^-, x_i^+]$, then $\dot{g}_i(\xi) \in [\sigma_i^-, \sigma_i^+]$, and $\dot{g}_i(x_i^*) \in [\sigma_i^-, \sigma_i^+]$.

Let $y_i(t) = x_i(t) - x_i^*$, then

$$\dot{y}_i(t) = -c_i y_i(t) + \sum_{j=1}^n a_{ij} f_j(y_j(t)) + \sum_{j=1}^n b_{ij} f_j(y_j(t-\tau)),$$
$$i = 1, 2, \ldots, n.$$

which can be rewritten in vector forms as follows:

$$\dot{y}(t) = -Cy(t) + Af(y(t)) + Bf(y(t-\tau)), \tag{9}$$

where $y = (y_1, \ldots, y_n)^T, f(y(t)) = (f_1(y_1(t)), \ldots, f_n(y_n(t)))^T$ and $f_i(y_i(t)) = g_i(x_i(t)) - g_i(x_i^*)$, where

$$\frac{f_i(y_i)}{y_i} = \frac{g_i(y_i + x_i^*) - g_i(x_i^*)}{y_i + x_i^* - y_i} = \dot{g}_i(\xi)$$

and $f_i(0) = 0$. Hence, if $x_i \in [x_i^-, x_i^+]$, then $\xi \in [x_i^-, x_i^+]$ and

$$\sigma_i^- \le \frac{f_i(y_i(t))}{y_i(t)} \le \sigma_i^+. \tag{10}$$

Denote

$$\Sigma_1 = diag(\sigma_1^+\sigma_1^-, \ldots, \sigma_n^+\sigma_n^-),$$
$$\Sigma_2 = diag\left(\frac{\sigma_1^+ + \sigma_1^-}{2}, \ldots, \frac{\sigma_n^+ + \sigma_n^-}{2}\right). \tag{11}$$

To prove our main theorem, we also need the following lemmas.

Lemma 2. *For any diagonal matrices $U = diag(u_1, \ldots, u_n) > 0$, $V = diag(v_1, \ldots, v_n) > 0$, if (10) holds, then*

$$\begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix}^T \begin{bmatrix} -U\Sigma_1 & U\Sigma_2 \\ U\Sigma_2 & -U \end{bmatrix} \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix} + \begin{bmatrix} y(t-\tau) \\ f(y(t-\tau)) \end{bmatrix}^T \begin{bmatrix} -V\Sigma_1 & V\Sigma_2 \\ V\Sigma_2 & -V \end{bmatrix} \begin{bmatrix} y(t-\tau) \\ f(y(t-\tau)) \end{bmatrix} \ge 0. \tag{12}$$

The proof can be seen in Liu, Wang, Serrano, and Liu (2007).

Lemma 3. *For real symmetric matrices $K > 0$, $M_i (i = 1, 2, 3, 4)$ with appropriate dimensions, then*

$$-\int_{t-\tau}^t \dot{y}^T(s) K \dot{y}(s) \mathrm{d}s \le \xi^T(t)[-\tau M^T K^{-1} M + M^T \hat{J} + \hat{J}^T M]\xi(t), \tag{13}$$

where

$$\xi(t) = [y^T(t), y^T(t-\tau), f^T(y(t)), f^T(y(t-\tau))]^T,$$
$$M = [M_1, M_2, M_3, M_4],$$
$$\hat{J} = [I, -I, 0, 0]^T.$$

Proof. Note the fact that

$$\begin{bmatrix} I & -M^T K^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} M^T K^{-1} M & M^T \\ M & K \end{bmatrix} \begin{bmatrix} I & -M^T K^{-1} \\ 0 & I \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \ge 0,$$

then one has

$$\begin{bmatrix} M^T K^{-1} M & M^T \\ M & K \end{bmatrix} \ge 0.$$

It follows that

$$\begin{aligned} 0 &\le \int_{t-\tau}^t \begin{bmatrix} \xi(t) \\ \dot{y}(s) \end{bmatrix}^T \begin{bmatrix} M^T K^{-1} M & M^T \\ M & K \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{y}(s) \end{bmatrix} \mathrm{d}s \\ &\le \tau \xi^T(t) M^T K^{-1} M \xi(t) + \xi^T(t) M^T \int_{t-\tau}^t \dot{y}(s) \mathrm{d}s \\ &\quad + \int_{t-\tau}^t \dot{y}^T(s) \mathrm{d}s M \xi(t) + \int_{t-\tau}^t \dot{y}(s) K \dot{y}(s) \mathrm{d}s \\ &\le \tau \xi^T(t) M^T K^{-1} M \xi(t) + \xi^T(t)[\hat{J}M + (\hat{J}M)^T]\xi(t) \\ &\quad + \int_{t-\tau}^t \dot{y}(s) K \dot{y}(s) \mathrm{d}s. \end{aligned}$$

Obviously, (13) holds. This completes the proof.

Lemma 4 (*Schur Complement, Boyd (1994)*). *Given constant symmetric matrices $\Omega_1, \Omega_2, \Omega_3$, where $\Omega_1 = \Omega_1^T$ and $\Omega_2 = \Omega_2^T > 0$, then*

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0.$$

Theorem 1. *For given τ, the equilibrium x^* in Λ is asymptotically stable, if Λ is positively invariant for (3) and there exist three symmetric matrices $P > 0, Q > 0, K > 0$, two diagonal matrices $U > 0, V > 0$, and M_i $(i = 1, \ldots, 4)$ with appropriate dimensions such that the LMI (14) holds*

$$\begin{bmatrix} \Omega + \hat{J}M + (\hat{J}M)^T & \tau M^T \\ \tau M & -\tau K \end{bmatrix} < 0, \tag{14}$$

where

$$\Omega = \begin{bmatrix} \Omega_{11} & 0 & PA + U\Sigma_2 - \tau C^T KA & PB - \tau C^T KB \\ * & -Q - V\Sigma_1 & 0 & V\Sigma_2 \\ * & * & -U + \tau A^T KA & \tau A^T KB \\ * & * & * & -V + \tau B^T KB \end{bmatrix},$$

$$\Omega_{11} = -(C^T P + PC) + Q - U\Sigma_1 + \tau C^T KC,$$

$M = [M_1, M_2, M_3, M_4]$, $\hat{J} = [I, -I, 0, 0]^T$ *and* Σ_1, Σ_2 *are defined in* (11). *Here,* $*$ *denotes the transpose of the corresponding upper diagonal elements of the matrix.*

Proof. Choose a Lyapunov–Krasovskii functional candidate as

$$V(t) = y^T(t)Py(t) + \int_{t-\tau}^{t} y^T(s)Qy(s)\mathrm{d}s + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{y}^T(s)K\dot{y}(s)\mathrm{d}s\mathrm{d}\theta, \tag{15}$$

where $P = P^T > 0, Q = Q^T > 0, K = K^T > 0$. Employing Lemmas 2 and 3, calculating the time-derivative of $V(t)$ along the trajectories,

$$\begin{aligned}
\dot{V}(t) &= 2y^T(t)P\dot{y}(t) + y^T(t)Qy(t) - y^T(t-\tau)Qy(t-\tau) \\
&\quad + \int_{-\tau}^{0} \dot{y}^T(t)K\dot{y}(t)\mathrm{d}\theta - \int_{-\tau}^{0} \dot{y}^T(t+\theta)K\dot{y}(t+\theta)\mathrm{d}\theta \\
&= 2y^T(t)P[-Cy(t) + Af(y) + Bf(y(t-\tau))] + y^T(t)Qy(t) \\
&\quad - y^T(t-\tau)Qy(t-\tau) + \tau[-Cy(t) + Af(y) + Bf(y(t-\tau))]^T \\
&\quad \times K[-Cy(t) + Af(y) + Bf(y(t-\tau))] - \int_{t-\tau}^{t} \dot{y}^T(s)K\dot{y}(s)\mathrm{d}s \\
&\le 2y^T(t)P[-Cy(t) + Af(y) + Bf(y(t-\tau))] + y^T(t)Qy(t) \\
&\quad - y^T(t-\tau)Qy(t-\tau) + \tau[-Cy(t) + Af(y) + Bf(y(t-\tau))]^T \\
&\quad \times K[-Cy(t) + Af(y) + Bf(y(t-\tau))] \\
&\quad + \xi^T(t)[-\tau M^T K^{-1} M + \hat{J}M + (\hat{J}M)^T]\xi(t) \\
&\quad + \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix}^T \begin{bmatrix} -U\Sigma_1 & U\Sigma_2 \\ U\Sigma_2 & -U \end{bmatrix} \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix} \\
&\quad + \begin{bmatrix} y(t-\tau) \\ f(y(t-\tau)) \end{bmatrix}^T \begin{bmatrix} -V\Sigma_1 & V\Sigma_2 \\ V\Sigma_2 & -V \end{bmatrix} \begin{bmatrix} y(t-\tau) \\ f(y(t-\tau)) \end{bmatrix} \\
&= \xi^T(t)[\Omega + \hat{J}M + (\hat{J}M)^T - \tau M^T K^{-1} M]\xi(t).
\end{aligned}$$

Employing Schur complement in Lemma 4, $\Omega + \hat{J}M + (\hat{J}M)^T - \tau M^T K^{-1} M < 0$ is guaranteed by LMI (14). If $\Omega + \hat{J}M + (\hat{J}M)^T - \tau M^T K^{-1} M < 0$, it yields $\dot{V}(t) < 0$ when $x_t \in \Lambda$, which implies x^* in the positive invariant set Λ is locally asymptotically stable and thus the proof is completed.

If we choose Lyapunov–Krasovskii functional (15) with $K = 0$, then the obtained stability criterion is delay-independent. The proof is similar to Theorem 1.

Theorem 2. *The equilibrium* x^* *in* Λ *is asymptotically stable, if* Λ *is positively invariant for* (3) *and there exist three symmetric matrices* $P > 0, Q > 0$, *two diagonal matrices* $U > 0, V > 0$ *with appropriate dimensions such that the LMI* (16) *holds*

$$\Omega = \begin{bmatrix} -(C^T P + PC) + Q - U\Sigma_1 & 0 & PA + U\Sigma_2 & PB \\ * & -Q - V\Sigma_1 & 0 & V\Sigma_2 \\ * & * & -U & 0 \\ * & * & * & -V \end{bmatrix} < 0 \tag{16}$$

where Σ_1, Σ_2 *are defined in* (11).

Remark 1. In the multistability criteria of Theorems 1 and 2, the choice of the positive invariant set Λ is much vital and flexible. On one hand, it is related to the stability discrimination of the equilibria. A smaller Λ can make the corresponding matrix Σ_1, Σ_2 more precise and the stability of the equilibria x^* in Λ will be determined more accurately. On the other hand, if the equilibria x^* is stable for the criterion above, then the the positive invariant set Λ is the attraction domain of x^*. If the inequality (14) or (16) holds on a larger Λ, then the attraction domain of x^* will be larger.

Remark 2. While considering stability for a large number of equilibria, we can take the different Σ_1, Σ_2 for each equilibria and solve every LMI to determine the stability for all equilibria. The computational complexity would be very high. For instance, if there exist 2^n equilibria, then we need to verify 2^n LMIs. In another way, we can set common Σ_1, Σ_2 for all equilibria in their neighborhood and examine just one LMI to verify whether all equilibria are stable. While the common Σ_1, Σ_2 would be more rough in general.

In the following, we will discuss the condition for Λ to be the positive invariant set.

$$(\mathrm{H}_3): \begin{cases} -c_i x_i^+ + \sum_{j=1}^{n} (a_{ij}^+ + b_{ij}^+) g_j(x_j^+) \\ \quad + \sum_{j=1}^{n} (a_{ij}^- + b_{ij}^-) g_j(x_j^-) + I_i < 0, \\ -c_i x_i^- + \sum_{j=1}^{n} (a_{ij}^+ + b_{ij}^+) g_j(x_j^-) \\ \quad + \sum_{j=1}^{n} (a_{ij}^- + b_{ij}^-) g_j(x_j^+) + I_i > 0, \end{cases} \qquad i = 1, 2, \ldots, n.$$

Here, $a_{ij}^+ = \max\{0, a_{ij}\}$, $a_{ij}^- = \min\{0, a_{ij}\}$, $b_{ij}^+ = \max\{0, b_{ij}\}$, $b_{ij}^- = \min\{0, b_{ij}\}$.

Lemma 5. *Under the condition* H_3, Λ *is positively invariant for* (3)

Proof. If Λ is not positively invariant, then there must exist a solution $x(t)$ with its initial value $\phi = (\phi_1, \ldots, \phi_n) \in \Lambda$, which leaves the region Λ first at time $t_0 > 0$. Without losing generality, assume x_i leaves $[x_i^-, x_i^+]$ first. Then $x_i(t_0) = x_i^+$, $\dot{x}_i(t_0) > 0$, or $x_i(t_0) = x_i^-$, $\dot{x}_i(t_0) < 0$.

Considering $x_i(t_0) = x_i^+$, from condition (H_3) we have

$$\begin{aligned}
\dot{x}_i(t_0) &= -c_i x_i(t_0) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t_0)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t_0 - \tau)) + I_i \\
&= -c_i x_i^+ + \sum_{j=1}^{n} (a_{ij}^+ + a_{ij}^-) g_j(x_j(t_0)) \\
&\quad + \sum_{j=1}^{n} (b_{ij}^+ + b_{ij}^-) g_j(x_j(t_0 - \tau)) + I_i \\
&\le -c_i x_i^+ + \sum_{j=1}^{n} (a_{ij}^+ + b_{ij}^+) g_j(x_j^+) + \sum_{j=1}^{n} (a_{ij}^- + b_{ij}^-) g_j(x_j^-) + I_i \\
&< 0,
\end{aligned}$$

It is inconsistent with $\dot{x}_i(t_0) > 0$. We can also have the same conclusion for the condition $x_i(t_0) = x_i^-$. Hence, $x(t)$ cannot leave Λ, and Λ is positively invariant for (3). This completes the proof.

Corollary 1. *For given τ, the equilibrium x^* in Λ is asymptotically stable, if condition (* H_3*) holds and there exist three symmetric matrices $P > 0, Q > 0, K > 0$, two diagonal matrices $U > 0, V > 0$, and M_i $(i = 1, \dots, 4)$ with appropriate dimensions such that the LMI (14) holds.*

Corollary 2. *The equilibrium x^* in Λ is asymptotically stable, if condition (* H_3*) holds and there exist three symmetric matrices $P > 0, Q > 0$, two diagonal matrices $U > 0, V > 0$ with appropriate dimensions such that the LMI (16) holds.*

Furthermore, under the conditions (H_1), (H_2), the regions Λ^α can be divided into two classes.

$$\Lambda_1 = \{\Lambda^\alpha \mid \exists\, i \le n, s.t.\, \alpha_i = \text{“}m\text{”}\},$$
$$\Lambda_2 = \{\Lambda^\alpha \mid \alpha_i = \text{“}l\text{” or “}r\text{”}\ \forall\, i \le n\}.$$

Λ_2 is composed of 2^n regions and Λ_1 is composed of $3^n - 2^n$ regions. If the equilibrium point $x^* \in \Lambda^\alpha$ and $\Lambda^\alpha \in \Lambda_1$, then x^* is usually unstable under the conditions (H_1), (H_2). Hence, we only consider the equilibrium points in Λ^α, where $\Lambda^\alpha \in \Lambda_2$.

Lemma 6. *Under the conditions (H_1), (H_2), if $b_{ii} > 0$ for any $i = \dots n$, then every $\Lambda^\alpha \in \Lambda_2$ is positively invariant for (3).*

Proof. If Λ^α is not positively invariant, then there must exist a solution $x(t)$ with its initial value $\phi = (\phi_1, \dots, \phi_n) \in \Lambda^\alpha$, which leaves the region Λ^α first at time $t_0 > 0$. Without losing generality, assume x_i leaves $[x_i^-, x_i^+]$ first and α_i = “l”. Then $x_i(t_0) = l_i^+$, $\dot{x}_i(t_0) > 0$, or $x_i(t_0) = l_i^-$, $\dot{x}_i(t_0) < 0$.

Considering $x_i(t_0) = l_i^+$, from condition (H_2) we have

$$\begin{aligned}\dot{x}_i(t_0) &= -c_i x_i(t_0) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t_0)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t_0 - \tau)) + I_i \\ &\le -c_i l_i^+ + (a_{ii} + b_{ii}) g_i(l_i^+) + \sum_{j=1, j\ne i}^{n} (|a_{ij}| + |b_{ij}|)\mu_j + I_i \\ &= h_i^+(l_i^+) \\ &= 0.\end{aligned}$$

It is inconsistent with $\dot{x}_i(t_0) > 0$. We can also have the same conclusion for the condition $x_i(t_0) = l_i^-$. Hence, $x(t)$ cannot leave Λ^α, and Λ^α is positively invariant for (3). This completes the proof.

Corollary 3. *Under the conditions (H_1), (H_2), take $\sigma_i^- = \min\{\dot{g}(l_i^-), \dot{g}(r_i^+)\}$, $\sigma_i^+ = \max\{\dot{g}(l_i^+), \dot{g}(r_i^-)\}$. For given $\tau \ge 0$, there are 2^n local asymptotically stable equilibria, if $b_{ii} > 0$ for any $i = \dots n$ and there exist three symmetric matrices $P > 0, Q > 0, K > 0$, two diagonal matrices $U > 0, V > 0$, and M_i $(i = 1, \dots, 4)$ with appropriate dimensions such that the LMI (14) holds.*

Proof. For any $\phi \in \Lambda^\alpha$, where $\Lambda^\alpha \in \Lambda_2$, we have

$$\min\{\dot{g}(l_i^-), \dot{g}(r_i^+)\} < \phi_i(\theta) < \max\{\dot{g}(l_i^+), \dot{g}(r_i^-)\}, \quad \forall\, \theta \in [-\tau, 0].$$

Hence, Corollary 3 can be derived directly from Theorem 1 and Lemma 6.

Corollary 4. *Under the conditions (H_1), (H_2), take $\sigma_i^- = \min\{\dot{g}(l_i^-), \dot{g}(r_i^+)\}$, $\sigma_i^+ = \max\{\dot{g}(l_i^+), \dot{g}(r_i^-)\}$. For given $\tau \ge 0$, there are 2^n local asymptotically stable equilibria, if $b_{ii} > 0$ for any $i = \dots n$ and there exist three symmetric matrices $P > 0, Q > 0, K > 0$, two diagonal matrices $U > 0, V > 0$ with appropriate dimensions such that the LMI (16) holds.*

Remark 3. In Theorem 3.2 (Cheng et al., 2007), the delay-independent multistability condition is obtained under the assumptions (H_1), (H_2), $b_{ii} > 0$ and

$$c_i > \sum_{j=1}^{n} \eta_j(|a_{ij}| + |b_{ij}|), \quad \text{for } i = 1, 2, \dots, n, \tag{17}$$

where $\max\{g_j'(l_j^+), g_j'(r_j^-)\} < \eta_j < \min\{g_j'(p_j), g_j'(q_j)\}$, $j = 1, \dots, n$. It is obvious that the signs of the weight connections and the delayed weight connections are neglected in the conditions above, that is to say, the differences between the neuronal excitatory and the inhibitory effects have been neglected. While, by using Lyapuonov–Krasovskii stability theorem and LMI method, our criteria avoid this problem.

4. Two examples

In this section, two examples are presented to illustrate both delay-independent and delay-dependent multistability results.

Example 1. Consider Example 6.4 in Cheng et al. (2007) as follows

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + 7g_1(x_1(t)) + 0.5g_2(x_2(t)) \\ \quad - 4g_1(x_1(t - \tau_{11})) + 0.5g_2(x_2(t - \tau_{12})), \\ \dot{x}_2(t) = -x_2(t) + 0.5g_1(x_1(t)) + 7g_2(x_2(t)) \\ \quad + 0.5g_1(x_1(t - \tau_{21})) - 4g_2(x_2(t - \tau_{22})), \end{cases} \tag{18}$$

where $g_1(x) = g_2(x) = \tanh(x)$. Here, we assume $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = \tau$. With the same computation in Cheng et al. (2007), conditions (H_1), (H_2) are satisfied, and $l_1^+ = l_2^+ = -1.8573$, $m_1^+ = m_2^+ = -0.5903$, $r_1^+ = r_2^+ = 3.9980$; $l_1^- = l_2^- = -3.9980$, $m_1^- = m_2^- = 0.5903$, $r_1^- = r_2^- = 1.8573$.

As the analysis and simulations in Example 6.4 Cheng et al. (2007), the equilibrium in $\Omega^{(l,l)}$, $\Omega^{(r,l)}$, $\Omega^{(l,r)}$, $\Omega^{(r,r)}$ is stable with $\tau_{11} < 0.08475$, $\tau_{22} < 0.08475$.

In fact, for each equilibria, we can make the matrix Σ_1, Σ_2 more precise. For the equilibrium (3.9973, 3.9973), choose the $[x_1^-, x_1^+] \times [x_2^-, x_2^+]$ in (8) as $[3.9, 4.1] \times [3.9, 4.1]$ and $\sigma_1^+ = \sigma_2^+ = 1.6376 \times 10^{-3}$, $\sigma_1^- = \sigma_2^- = 1.0980 \times 10^{-3}$,

$$\Sigma_1 = \begin{pmatrix} 1.798 \times 10^{-4} & 0 \\ 0 & 1.798 \times 10^{-4} \end{pmatrix},$$
$$\Sigma_2 = \begin{pmatrix} 1.367 \times 10^{-3} & 0 \\ 0 & 1.367 \times 10^{-3} \end{pmatrix}.$$

For the equilibrium $(1.9150, -1.9150)$, choose the $[x_1^-, x_1^+] \times [x_2^-, x_2^+]$ in (8) as $[1.91, 1.92] \times [1.91, 1.92]$ and $\sigma_1^+ = \sigma_2^+ = 8.2394 \times 10^{-2}$, $\sigma_1^- = \sigma_2^- = 8.3987 \times 10^{-2}$,

$$\Sigma_1 = \begin{pmatrix} 6.920 \times 10^{-3} & 0 \\ 0 & 6.920 \times 10^{-3} \end{pmatrix},$$
$$\Sigma_2 = \begin{pmatrix} 8.319 \times 10^{-2} & 0 \\ 0 & 8.319 \times 10^{-2} \end{pmatrix}.$$

And, it can be verified that condition (H_3) holds for the two equilibria. From Corollary 2, they are all asymptotically stable for any $\tau \ge 0$. To the other two equilibria, we can have the same conclusion. Different from the simulation in Cheng et al. (2007), Figs. 5 and 6 depict the dynamics with $\tau = 10$, and the attraction basins for $(1.9150, -1.9150)$ and $(-1.9150, 1.9150)$ are really small.

Example 2. Consider the neural network as follows

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + 4g_1(x_1(t)) - 2g_2(x_2(t)) \\ \quad + 3g_1(x_1(t - \tau)) + 2.8g_2(x_2(t - \tau)), \\ \dot{x}_2(t) = -x_2(t) - 2g_1(x_1(t)) + 4g_2(x_2(t)) \\ \quad + 2.8g_1(x_1(t - \tau)) + 3g_2(x_2(t - \tau)), \end{cases} \tag{19}$$

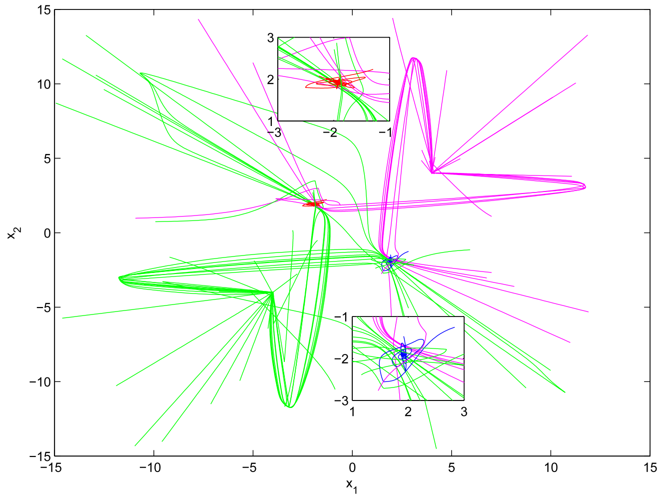

Fig. 5. Phase plot of (x_1, x_2) in Example 1 with $\tau = 10$. The subfigures plot the dynamic behaviors near the equilibria $(1.9150, -1.9150)$ and $(-1.9150, 1.9150)$. The trajectories in the same color converge to the same stable equilibrium point.

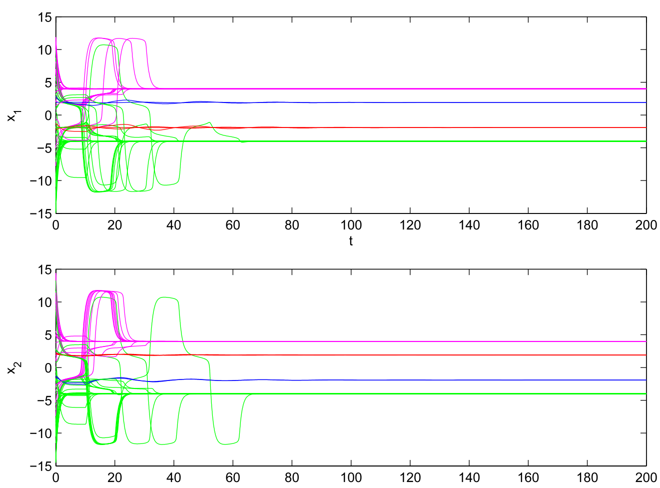

Fig. 6. Response of x_1, x_2 in Example 1 with $\tau = 10$.

where $g_1(x) = g_2(x) = \tanh(x)$ and $\tau = 10$. Direct computation gives

$$h_1^+(\xi) = h_2^+(\xi) = -\xi + 7\tanh(\xi) + 4.8,$$
$$h_1^-(\xi) = h_2^-(\xi) = -\xi + 7\tanh(\xi) - 4.8.$$

The parameters satisfy the criterion in Corollary 3:

$$\begin{aligned}
\text{Condition}(\mathrm{H}_1): \quad & 0 < \frac{c_1}{a_{11} + b_{11}} = \frac{1}{7} < 1, \\
& 0 < \frac{c_2}{a_{22} + b_{22}} = \frac{1}{7} < 1; \\
\text{Condition}(\mathrm{H}_2): \quad & p_1 = p_2 = -1.6283, \qquad q_1 = q_2 = 1.6283, \\
& h_1^+(p_1) = h_2^+(p_2) = -0.0524 < 0, \\
& h_1^-(q_1) = h_2^-(q_2) = 0.0524 > 0.
\end{aligned}$$

And, $l_1^+ = l_2^+ = -1.8838$, $m_1^+ = m_2^+ = -1.4050$, $r_1^+ = r_2^+ = 11.8000$; $l_1^- = l_2^- = -11.8000$, $m_1^- = m_2^- = 1.4050$, $r_1^- = r_2^- = 1.8838$. Set

$$\Sigma_1 = \begin{pmatrix} 1.989 \times 10^{-9} & 0 \\ 0 & 1.989 \times 10^{-9} \end{pmatrix},$$

$$\Sigma_2 = \begin{pmatrix} 0.0442 & 0 \\ 0 & 0.0442 \end{pmatrix},$$

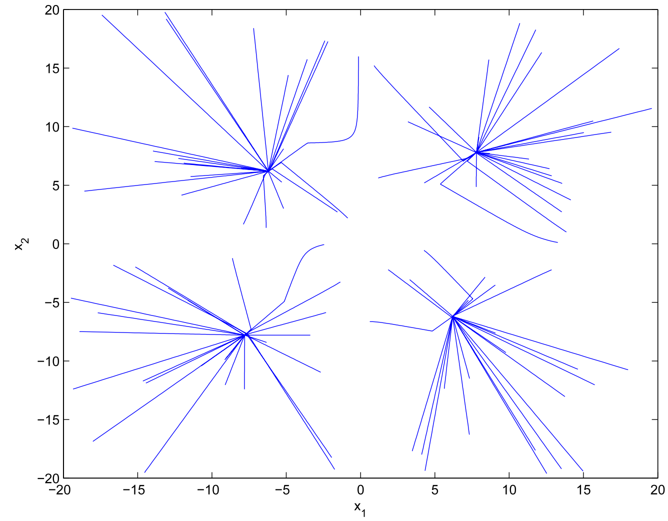

Fig. 7. Phase plot of (x_1, x_2) in Example 2 with $\tau = 10$.

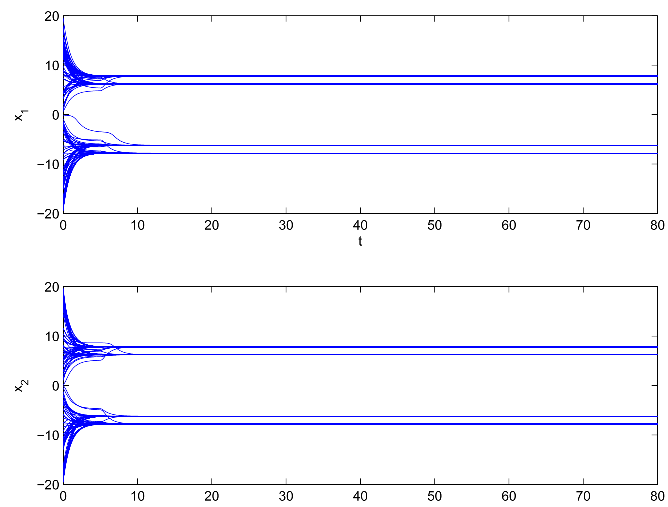

Fig. 8. Time response of x_1, x_2 in Example 2 with $\tau = 10$.

thus we can easily verify that the LMI (14) is satisfied. From Corollary 3, there exists 2^n stable equilibria. The parameters herein do not satisfy the criterion (17) for theory in Cheng et al. (2007):

$$\begin{aligned}
&\eta_1(|a_{11}| + |b_{11}|) + \eta_2(|a_{12}| + |b_{12}|) \\
&\quad > \max\{g_1'(l_1^+), g_1'(r_1^-)\}(|a_{11}| + |b_{11}|) \\
&\qquad + \max\{g_2'(l_2^+), g_2'(r_2^-)\}(|a_{12}| + |b_{12}|) \\
&\quad = 1.0420 > 1 \\
&\quad = c_1, \\
&\eta_1(|a_{21}| + |b_{21}|) + \eta_2(|a_{22}| + |b_{22}|) \\
&\quad > \max\{g_1'(l_1^+), g_1'(r_1^-)\}(|a_{21}| + |b_{21}|) \\
&\qquad + \max\{g_2'(l_2^+), g_2'(r_2^-)\}(|a_{22}| + |b_{22}|) \\
&\quad = 1.0420 > 1 \\
&\quad = c_2,
\end{aligned}$$

which demonstrate the assertion in Remark 3. The dynamics of (19) are illustrated in Figs. 7 and 8.

5. Coexistence of equilibria and limited cycles

In this section, we consider a special case to show the coexistence of equilibria and limited cycles.

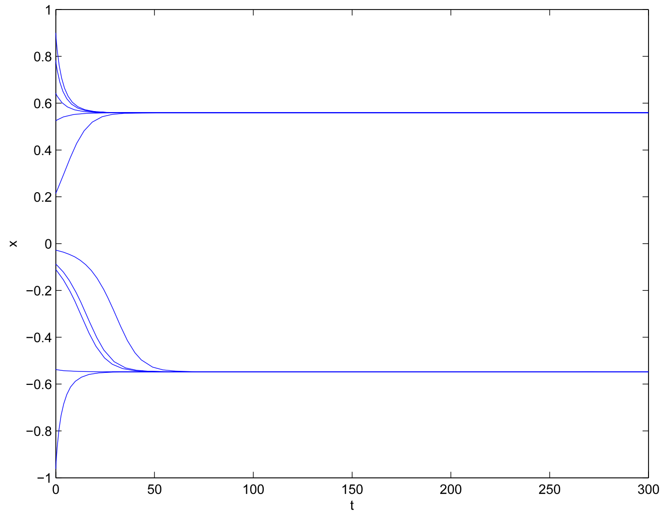

Fig. 9. Dynamic behaviors for Eq. (20) with $\tau = 0$.

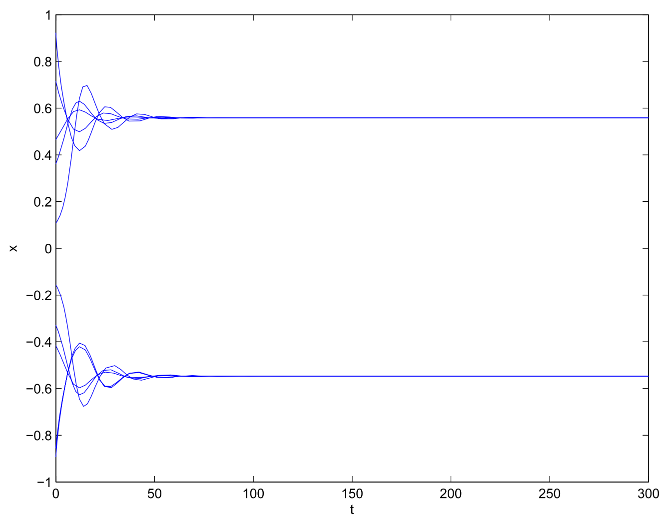

Fig. 10. Dynamic behaviors for Eq. (20) with $\tau = 6$.

5.1. A single neuron

Consider a single neuron as follows,

$$\dot{x}(t) = -x(t) + ag(x(t)) + bg(x(t-\tau)), \tag{20}$$

where $a = 1.3, b = -0.2$, and $g(x) = \tanh(x)$ and conditions (H_1), (H_2) are satisfied. Hence, there exist 3 equilibria for Eq. (20). Solve the stationary equations of Eq. (20), as follows,

$$-x + 1.1\tanh(x) = 0,$$

we can find the 3 equilibria as $x_0 = 0$, $x_1 = 0.5532$, $x_2 = -0.5532$.

Consider the linear system for Eq. (20) at x_0

$$\dot{x}(t) = 0.3x(t) - 0.2x(t-\tau),$$

where the characteristic equation is

$$h_0(\lambda) = \lambda - 0.3 + 0.2e^{-\tau\lambda} = 0.$$

If $\lambda = 0$, then $h_0(\lambda) < 0$; if $\lambda = 0.3$, then $h_0(\lambda) > 0$. Hence, there exists a real root $\lambda \in [0, 0.3]$ for characteristic equation $h_0(\lambda) = 0$. The origin is unstable for any $\tau > 0$.

Consider the linear system for Eq. (20) at x_1

$$\begin{aligned}\dot{x}(t) &= -x(t) + a\tanh'(x_1)x(t) + b\tanh'(x_1)x(t-\tau)\\ &= -x(t) + 0.9711x(t) - 0.1494x(t-\tau),\end{aligned}$$

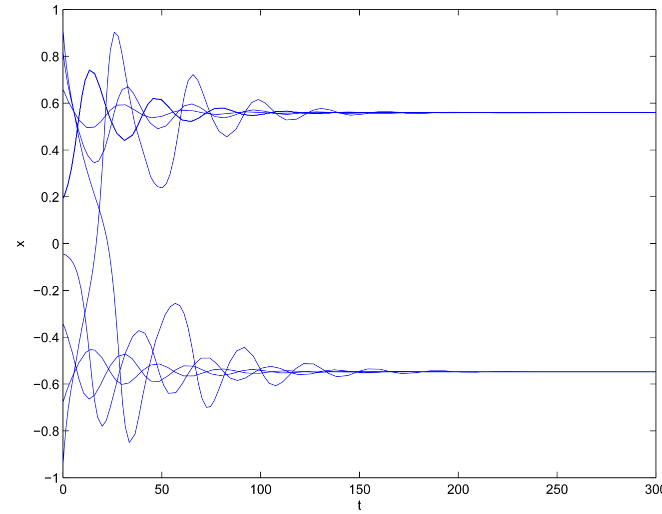

Fig. 11. Dynamic behaviors for Eq. (20) with $\tau = 8$.

where the characteristic equation is

$$h_1(\lambda) = \lambda + 0.0289 + 0.1494e^{-\tau\lambda} = 0.$$

If $\tau < 12.0445$, there is no root for $h_1(\lambda) = 0$ with positive real part. Hence, x_1 is stable for $\tau < 12.0445$. For symmetry, we can gain the same conclusion for x_2. Note that, applying Theorem 1, we can get that x_1, x_2 are asymptotically stable for $\tau < 10.5341$. The dynamic behaviors for Eq. (20) are illustrated in Figs. 9–14 with different τ.

From the simulations in Figs. 9–14, if $\tau \leq 8$, there exists two stable equilibria for Eq. (20). As $\tau = 9$, the two stable equilibria are preserved and a stable limit cycle appears. As τ increasing, the stable limit cycle holds, but the two equilibria become unstable for $\tau > 12.0445$. And, Fig. 14 shows the dynamics for $\tau = 13$.

Remark 4. Different from Hopf bifurcation (see Hassard et al., 1981; Song et al., 2005; Zhu & Huang, 2007), the stable limit cycle emerges before the stability of all equilibrium points changed. Compared with other coexistence phenomenon for stable patterns, such as Campbell et al. (2006), the coexistence of stable asynchronous limit cycles and stable synchronous equilibria has been found in the single neuron model in this article.

5.2. Two coupled neurons

As illustrated above, we found the coexistence of stable equilibria and a stable limit cycle for a single neuron. Consider neural network, which is weakly coupled by two neurons, as follows

$$\begin{cases}\dot{x_1}(t) = -x_1(t) + ag(x_1(t)) + bg(x_1(t-\tau)) + cg(x_2(t)),\\ \dot{x_2}(t) = -x_2(t) + ag(x_2(t)) + bg(x_2(t-\tau)) + cg(x_1(t)),\end{cases} \tag{21}$$

where $a = 1.3, b = -0.2, g(x) = \tanh(x)$, which is the same with Eq. (20), and $c = 0.016$. From the stationary equations,

$$\begin{cases}-x_1 + 1.1\tanh(x_1) + 0.016\tanh(x_2) = 0,\\ -x_2 + 1.1\tanh(x_2) + 0.016\tanh(x_1) = 0,\end{cases}$$

Eq. (21) has 9 equilibrium points as:

$$\begin{aligned}&x^{(r,l)} = (0.5062, -0.5062), \quad x^{(r,m)} = (0.5458, -0.0816),\\ &\quad x^{(r,r)} = (0.5968, 0.5968),\\ &x^{(m,l)} = (0.0816, -0.5458), \quad x^{(m,m)} = (0, 0),\\ &\quad x^{(m,r)} = (-0.0816, 0.5458),\\ &x^{(l,l)} = (-0.5968, -0.5968), \quad x^{(l,m)} = (-0.5458, 0.0816),\\ &\quad x^{(l,r)} = (-0.5062, 0.5062).\end{aligned}$$

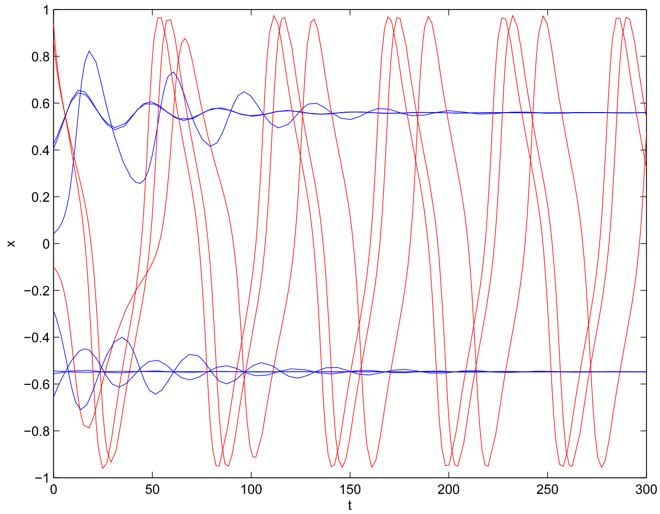

Fig. 12. Dynamic behaviors for Eq. (20) with $\tau = 9$.

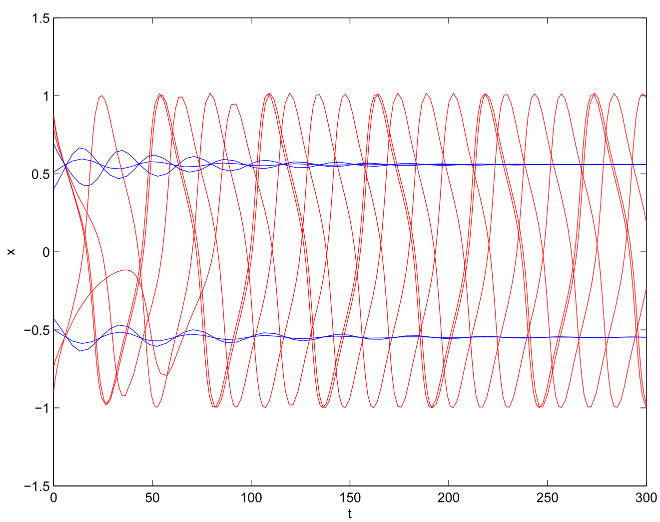

Fig. 13. Dynamic behaviors for Eq. (20) with $\tau = 9.6$.

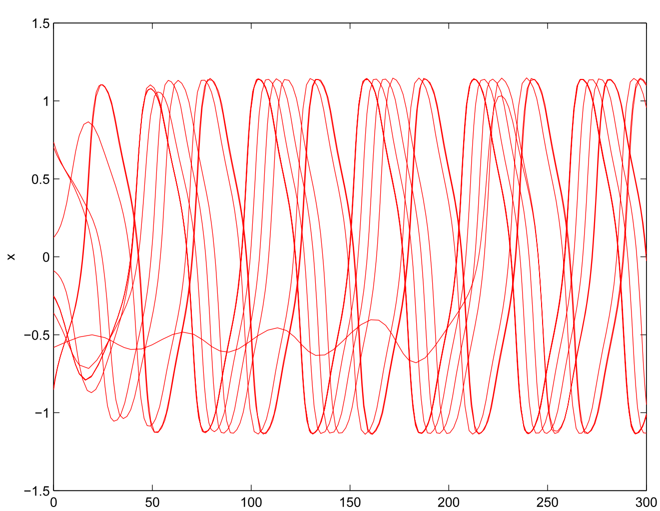

Fig. 14. Dynamic behaviors for Eq. (20) with $\tau = 13$.

By applying Theorem 1, we can get that, $x^{(r,l)}$ and $x^{(l,r)}$ are stable for $\tau < 8.8305$; $x^{(r,r)}$ and $x^{(l,l)}$ are stable for $\tau < 11.7244$.

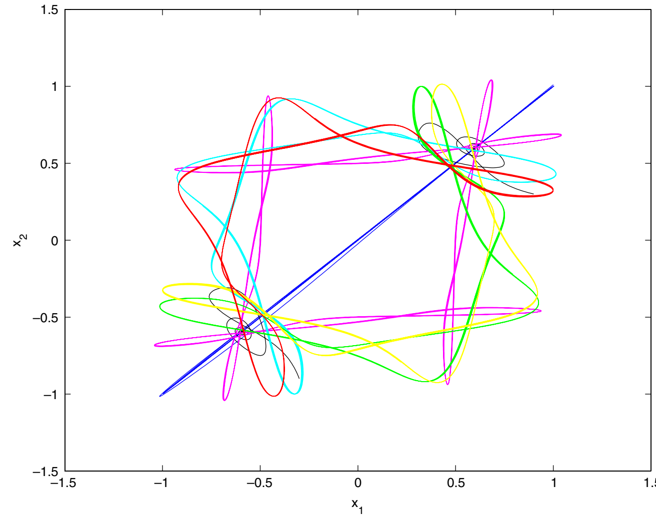

Fig. 15. Stable patterns of Eq. (21) with $\tau = 9.6$.

Consider $\tau = 9.6$, for a single neuron, there exist threepatterns, as two stable equilibria and a stable limit cycle. If the coupling strength between the two neurons is small enough, there should be 3^2 stable patterns, which are compound of the different stable patterns of the single neuron.

As the simulations in Fig. 15, the stable patterns are depicted by different colors. From the computation above, the equilibria $x^{(r,r)}$ and $x^{(l,l)}$ are stable, which are drawn in black. It appears that the equilibria $x^{(r,l)}$ and $x^{(l,r)}$ are unstable. Since the connection from the two neurons $c = 0.016$ is so weak, the limit cycle, compounded of the limit cycle for the single neuron, is still stable as depicted in blue in Fig. 15. And, the stability of the four limit cycles, compound with the limit cycle and the stable equilibria, are also preserved, which are depicted in magenta. Besides the compound stable patterns, four other stable limit cycles emerge in the neural network, which are depicted in cyan, yellow, green, and red correspondingly in Fig. 15. Hence, it could be found that two stable equilibrium points and nine limit cycles coexist in the neural network (21). There are 11 stable patterns in all, which is larger than $3^2 = 9$. If this type of delay-dependent neural networks are applied in associative memory storage or pattern recognition, then the memory capability and the memory mode will be improved, and they can be adjusted by delay.

Note that, different from the stable periodic orbits evoked by periodic external inputs, they are constants for all the external inputs in Eq. (21). In Cheng et al. (2007) and Zeng and Wang (2006), under the periodic external inputs, all the stable patterns are limit cycles, and there is no stable equilibrium in the neural networks. However, in Eq. (21), there is a coexistence of both limit cycles and equilibria.

6. Conclusion

In this article, the delay-dependent multistability of neural networks is studied. By utilizing LMI approach, both delay-dependent and delay-independent criteria for multistability are obtained. And, the stability of the equilibria can be verified more flexibly by the given criteria. Compared with the existing multistability results, the criteria obtained in this article are less conservative. The simulation results show the validity of our criteria.

Furthermore, by using the obtained criteria and simulations, a special example is investigated. It is found that there could be three stable patterns for a single neuron. If the neurons are coupled into a neural network,then besides the compound of the stable patterns form the single neurons, some new stable patterns would emerge, in which delay really enrich the dynamic behaviors of the neural networks. However, we could only prove the stability of the equilibria in this article. The existence and stability of the limit cycles are still open problems. Also, the mechanisms for the emergence of the new type of stable patterns are really unclear.

References

Boyd, S. P. (1994). *Linear matrix inequalities in system and control theory*. Society for Industrial & Applied.

Campbell, S. A., Ncube, I., & Wu, J. (2006). Multistability and stable asynchronous periodic oscillations in a multiple-delayed neural system. *Physica D*, *214*(2), 101–119.

Cao, J., Ho, D. W. C., & Huang, X. (2007). LMI-based criteria for global robust stability of bidirectional associative memory networks with time delay. *Nonlinear Analysis, TMA*, *66*(7), 1558–1572.

Cao, J., & Li, X. (2005). Stability in delayed cohen-grossberg neural networks: LMI optimization approach. *Physica D*, *53*(1–2), 54–65.

Cheng, C., Lin, K. H., & Shih, C. W. (2006). Multistability in recurrent neural networks. *SIAM Journal on Applied Mathematics*, *66*, 1301–1320.

Cheng, C. Y., Lin, K. H., & Shih, C. W. (2007). Multistability and convergence in delayed neural networks. *Physica D*, *225*, 61–74.

Chua, L. O. (1998). *CNN: A paradigm for complexity*. River Edge, NJ: World Scientific.

Cichocki, A. (2002). *Adaptive blind signal and image processing: Learning algorithms and applications*. New York, NY, USA: John Wiley & Sons, Inc..

Cochocki, A., & Unbehauen, R. (1993). *Neural networks for optimization and signal processing*. New York, NY, USA: John Wiley & Sons, Inc.

Cohen, M. A. (1992). The construction of arbitrary stable dynamics in nonlinear neural networks. *Neural Networks*, *225*(5), 83–103.

Forti, M., Nistri, P., & Quincampoix, M. (2004). Generalized neural network for nonsmooth nonlinear programming problems. *IEEE Transactions Circuits and Systems Part I*, *51*(9), 1741–1754.

Forti, M., & Tesi, A. (1995). New conditions for global stability of neural networks withapplication to linear and quadratic programming problems. *IEEE Transactions Circuits and Systems Part I*, *42*(7), 354–366.

Foss, J., Longtin, A., Mensour, B., & Milton, J. (1996). Multistability and delayed recurrent loops. *Physica Review Letters*, *76*, 708–711.

Hassard, B.D., Kazarinoff, N.D., & Wan, Y.H. (1981). Theory and applications of Hopf bifurcation. London England: Cambridge.

He, Y., Wang, Q. G., Wu, M., & Lin, C. (2006). Delay-dependent state estimation for delayed neural networks. *IEEE Transactions Neural Networks*, *17*(4), 1077–1081.

He, Y., & Wu, M. (2006). An improved global asymptotic stability criterion for delayed cellular neural networks. *IEEE Transactions Neural Networks*, *17*, 250–253.

Hopfield, J. (1984). Neurons with graded response have collective computational properties like those of two-state neurons. *Proceedings of the National Academy of Sciences USA*, *81*, 3088–3092.

Karhunen, J., Hyvarinen, A., Vigario, R., Hurri, J., & Oja, E. (1997). Applications of neural blind separation to signal and image processing. *Proceedings of ICASSP*, *1*, 131–134.

Li, C. D., & Chen, J. Y. (2007). A new criterion for global robust stability of interval neural networks with discrete time delays. *Chaos, Solitions and Fractals*, *31*, 561–570.

Liu, Y., Wang, Z., Serrano, A., & Liu, X. (2007). Discrete-time recurrent neural networks with time-varying delays: Exponential stability analysis. *Physics Letters A*, *362*(5–6), 480–488.

Lou, X. Y., & Cui, B. T. (2006). On the global robust asymptotic stability of bam neural networks with time-varying delays. *Neurocomputing*, *70*, 273–279.

Lu, W., & Chen, T. (2004). Synchronization of coupled connected neural networks with delays. *IEEE Transactions Circuits and Systems Part I*, *51*(12), 2491–2503.

Morita, M. (1993). Associative memory with non-monotone dynamics. *Neural Networks*, *6*, 115–126.

Singh, V. (2006). New global robust stability results for delayed cellular neural networks based on norm-bounded uncertainties. *Chaos, Solitons and Fractals*, *30*, 1165–1171.

Song, Y., Han, M., & Wei, J. (2005). Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays. *Physica D*, *200*(3–4), 185–204.

Wang, Z, Ho, D. W. C., & Liu, X. (2005). State estimation for delayed neural networks. *IEEE Transactions on Neural Networks*, *16*(1), 279–284.

Wang, Z, Shu, H., Liu, Y., Ho, D. W. C., & Liu, X. (2006). Robust stability analysis of generalized neural networks with discrete and distributed time delays. *Chaos, Solitons and Fractals*, *30*, 886–896.

Xia, Y., Leung, H., & Bosse, E. (2002). Neural data fusion algorithms based on a linearly constrained leastsquare method. *IEEE Transactions Neural Networks*, *13*(2), 320–329.

Xu, S. Y., & Lam, J. (2006). A new lmi condition for delay-dependent asymptotic stability of delayed hopfield neural networks. *IEEE Transactions Circuits and Systems Part II: Express Briefs*, *53*, 230–235.

Yu, W., & Cao, J. (2007). Synchronization control of stochastic delayed neural networks. *Physica A*, *373*, 252–260.

Zeng, Z., Huang, D., & Wang, Z. (2005). Memory pattern analysis of cellular neural networks. *Physics Letters A*, *342*, 114–128.

Zeng, Z., & Wang, J. (2006). Multiperiodicity and exponential attractivity evoked by periodic external inputs in delayed cellular neural networks. *Neural Computation*, *18*, 848–870.

Zeng, Z., Wang, J., & Liao, X. (2004). Stability analysis of delayed cellular neural networks described using cloning templates. *IEEE Transactions Circuits and Systems Part I*, *51*, 2313–2324.

Zhu, H., & Huang, L. (2007). Stability and bifurcation in a tri-neuron network model with discrete and distributed delays. *Applied Mathematics and Computation*, *188*(2), 1742–1756.