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The minority's success under majority rule[☆]

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ABSTRACT

In this paper, we focus on the effect of a network's structure on the process of opinion formation. Emphasis is placed on the minority's opinion evolution in a community structured network, where the majority rule is applied to govern the evolution. A model is developed for theoretical analysis using the mean field method. In this model, the connections are dense in the community, but sparse outside. A bifurcation diagram can thus be constructed, which is also verified through experimental study. The phase transition in the evolution is also investigated. In addition, a further investigation shows that a larger group size would bring more advantage to the minority.

1. Introduction

In this paper, we investigate the effect of the community structure in opinion evolution by majority rule. The basic fundamental of majority rule is that the opinion a social group has on a topic changes when the individuals meet and discuss in small groups. After discussion, the majority opinion is adopted by all the members of the discussion group. As pointed out in Ref. [1], this simple decision-making process leads to rich collective behavior. In general, the initial majority will determine the final state under this rule [1]. However this is not always the case. In Refs. [2,3], Galam employed a physical principle called the "inertia principle" for the opinion propagation of the minority. The effect of inflexible nodes is also discussed in Ref. [4]. As a result, a perfect equality is obtained or the initial minority wins under certain conditions.

In social network modeling, various networks have gradually been proposed. From the simplest full connected networks, Euclidean geometry in one, two or more dimensions [5,6], to very advanced network topologies [7–13], the network models are much better at representing the actual relationships between people and groups. It is clear that the evolution of opinion formation is deeply dependent on the topologies of the networks. In Ref. [6], the dynamics of the social impact models are compared in four simple topologies. Moreover, the introduction of the small world [7–9] and scale free networks [10] has also led to interesting results in social physics [14]. As pointed out in Ref. [14], the existence of small-world type shortcuts allows information to spread quickly through the society. Local opinions are capable of influencing distant parts immediately. In scale free networks, the hubs—highly connected individuals—can influence a lot of other members in the society. The ease or difficulty of convincing such highly connected agents is often crucial to achieving consensus. Furthermore, an interesting problem that asks whether the dynamics taking place on a network controls the network structure or the structure controls the dynamics is investigated in Ref. [11].

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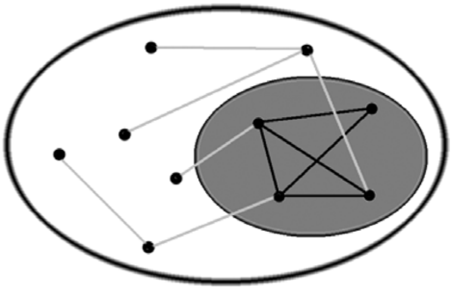

Fig. 1. A schematic representation of a network with community structure. In this network there is just one community of densely connected vertices (solid lines), with a much lower density of connections (gray lines) between them.

Including the small-world property, heavy-tailed degree distributions and others, a number of different characteristics have been found to occur commonly in real networks. Another common characteristic is community structure [12]. Based on common location, interests, occupation etc., social networks often include community groups. Community structure refers to the occurrence of groups of nodes in a network that are more densely connected internally than with the rest of the network. Ref. [13] analyzes the opinion dynamics in a topology consisting of two coupled fully connected networks to mimic the existence of communities in social networks. A transition behavior takes place at the value of the interconnectivity parameter. Moreover, algorithms to find communities become an interesting topic in physics, sociology and computer society [12,15].

In this paper, to understand the mechanism of the community structure in the opinion dynamics, we focus on the following two questions:

(i) Could the community structure help the minority in its propagation? If yes, how?
(ii) What factors will affect the evolution result?

For simplicity, a single community model is proposed. Different from the assumption of social inertia [3] or inflexible nodes [4] in the process of opinion formation, we claim that the minority shows no more aggressiveness or persuasiveness than the others. Also, there is no temperature-like parameter as in the social impact model investigated in Ref. [16]. Here we only focus on the effect of topologies on the opinion dynamics.

The rest of the paper is organized as follows. In Section 2, the structure of the network and the interactions between opinions are introduced. The dynamics of the evolution for $G = 3$ are analyzed in Section 3, and the condition for $G > 3$ is discussed in Section 4. Finally, the conclusions are summarized in Section 5.

2. Model description

Consider a network with N nodes, with opinion α or β. The community is composed of the first M nodes. At each time step t, any two randomly chosen nodes have an edge with the probability p_1, if there is at least one of the two nodes outside the community, and the link probability for any two nodes in the community is p_2. It is worth noting that the topology is not static. At each time step, any two nodes in the network has an edge with a certain probability, but the values of the link probabilities p_1, p_2 are invariant. Here $p_2 \geq p_1$ means a more dense internal connection of the community, and $M < \frac{1}{2}N$ means the population of the community is a minority in the network. A schematic representation of a network is depicted in Fig. 1. For convenience, we denote the fraction of the minority $\nu = M/N$ and the link probability ratio $\omega = p_1/p_2$.

Compared with the global minority for the community, the opinion evolution is governed by the local majority rule. At each time step t, a group of G fully connected nodes is randomly selected. G is the group size. The majority opinion will be adopted by all the nodes in the group. If there exists a tie in the group, then all the nodes will remain unchanged. For example, in the case of group size $G = 4$, the update rules are written as

$$\begin{aligned} \alpha\alpha\alpha\alpha, \alpha\alpha\alpha\beta &\rightarrow \alpha\alpha\alpha\alpha \\ \alpha\beta\beta\beta, \beta\beta\beta\beta &\rightarrow \beta\beta\beta\beta \\ \alpha\alpha\beta\beta &\rightarrow \alpha\alpha\beta\beta. \end{aligned}$$

3. The analysis for $G = 3$

3.1. Mean-field analysis

For the sake of clarity, we consider a fully connected network with $G = 3$ first. As the network size $N \rightarrow \infty$, the average number of nodes with opinion α evolves as

$$A(t+1) = A(t) + 3a^2(1-a) - 3a(1-a)^2$$

Table 1
Symbols and their meanings.

Symbols	Meanings
A_i	Average number of nodes α inside the community
A_o	Average number of nodes α outside the community
$a_i = A_i/M$	Average fraction of nodes α inside the community
$a_o = A_o/(N-M)$	Average fraction of nodes α outside the community
$b_i = 1 - a_i$	Average fraction of nodes β inside the community
$b_o = 1 - a_o$	Average fraction of nodes β outside the community
$R_{i,(x,y)}$	Contribution for the selection (x, y) to the evolution of A_i
$R_{o,(x,y)}$	Contribution for the selection (x, y) to the evolution of A_o

where $a = A/N$ is the average fraction of nodes with opinion α. The probabilities for the selections (α, α, β), (α, β, β) are $C_3^2 a^2(1-a)$, $C_3^1 a(1-a)^2$ correspondingly, where $C_k^j = \frac{k!}{j!(k-j)!}$. So the contribution to evolution of A is

$$R = 3a^2(1-a) - 3a(1-a)^2.$$

By employing the community structure, the situation is complicated. At each time step, a fully coupled triplet is randomly selected with the probability $P_{(x,y)}$. The value of $P_{(x,y)}$ is the probability that the selected triplet is composed of x nodes inside the community and y nodes outside, where $x + y = 3$. Four possible triplets may be selected with the probability.

$$P_{(0,3)} = C_3^0 P_o P_o P_o p_1^3 = (1-\nu)^3 w^3 p_2^3,$$
$$P_{(1,2)} = C_3^1 P_i P_o P_o p_1^3 = 3\nu(1-\nu)^2 w^3 p_2^3,$$
$$P_{(2,1)} = C_3^2 P_i P_i P_o p_1^2 p_2 = 3\nu^2(1-\nu) w^2 p_2^3,$$
$$P_{(3,0)} = C_3^3 P_i P_i P_i p_2^3 = \nu^3 p_2^3,$$

where $P_i = M/N = \nu$ is the probability that the chosen node is in the community, similarly $P_o = (N-M)/N = 1-\nu$ represents the probability that the chosen node is outside the community. Some other notations are listed in Table 1.

For the group size $G = 3$, the contributions $R_{i,(x,y)}$ and $R_{o,(x,y)}$ for the four possible possibilities are as follows,

$$\begin{aligned} R_{i,(3,0)} &= 3a_i^2 b_i - 3a_i b_i^2 & R_{i,(3,0)} &= 0 \\ R_{i,(2,1)} &= 2a_i b_i a_o - 2a_i b_i b_o & R_{o,(2,1)} &= a_i^2 b_o - b_i^2 a_o \\ R_{i,(1,2)} &= a_o^2 b_i - b_o^2 a_i & R_{o,(1,2)} &= 2a_o b_o a_i - 2a_o b_o b_i \\ R_{i,(0,3)} &= 0 & R_{o,(0,3)} &= 3a_o^2 b_o - 3a_o b_o^2. \end{aligned}$$

Thus A_i, A_o obey the master equations

$$A_i(t+1) = A_i(t) + P_{(3,0)} R_{i,(3,0)} + P_{(2,1)} R_{i,(2,1)} + P_{(1,2)} R_{i,(1,2)} + P_{(0,3)} R_{i,(0,3)}$$
$$A_o(t+1) = A_o(t) + P_{(3,0)} R_{o,(3,0)} + P_{(2,1)} R_{o,(2,1)} + P_{(1,2)} R_{o,(1,2)} + P_{(0,3)} R_{o,(0,3)}.$$

As $N \to \infty$, we also consider evolution for continuous time. The corresponding model can be given by the following differential equations:

$$\begin{cases} \dfrac{da_i}{dt} = \dfrac{p_2^3}{\nu}[\nu^3 \times (3a_i^2 b_i - 3a_i b_i^2) + 3\nu^2(1-\nu)w^2 \times (2a_i b_i a_o - 2a_i b_i b_o) + 3\nu(1-\nu)^2 w^3 \times (a_o^2 b_i - b_o^2 a_i)] \\ \dfrac{da_o}{dt} = \dfrac{p_2^3}{1-\nu}[3\nu^2(1-\nu)w^2 \times (a_i^2 b_o - b_i^2 a_o) + 3\nu(1-\nu)^2 w^3 \times (2a_o b_o a_i - 2a_o b_o b_i) \\ \qquad + (1-\nu)^3 w^3 \times (3a_o^2 b_o - 3a_o b_o^2)]. \end{cases} \tag{1}$$

It is clear that $(0, 0)$, $(1, 1)$, $(\frac{1}{2}, \frac{1}{2})$ are the equilibria of Eq. (1). And numerical calculations indicate that there are no other equilibria for Eq. (1). The linearized method also verifies that the equilibria $(0, 0)$, $(1, 1)$ are stable, and $(\frac{1}{2}, \frac{1}{2})$ is a saddle point. Hence with the initial state $(a_i(0), a_o(0)) = (1, 0)$, the network will converge to a consensus finally. One of the opinions α and β would vanish. And no metastable condition [13,16] could be observed in this model.

In Eq. (1), it could be found that the evolution results are not affected by p_2, but the link probability ratio $\omega = p_1/p_2$. The value of p_2 only affects the convergence time in the evolution process. Hence, in the following simulations, we always assume $p_2 = 1$.

3.2. Computer simulations

For group size $G = 3$, simulations are compared with the theoretical results in Fig. 2(a). The critical points, defined by ω_c, decline with decreasing ν. If $\omega < \omega_c$ (the shaded area in Fig. 2(a)), the initial minority is able to conquer the whole network. So the community structured topology will help the minority in its evolution. As $\nu = M/N < \frac{1}{2}$, the community is the minority in the network. However with a certain value of $\omega < 1$, the community structure makes the connections inside

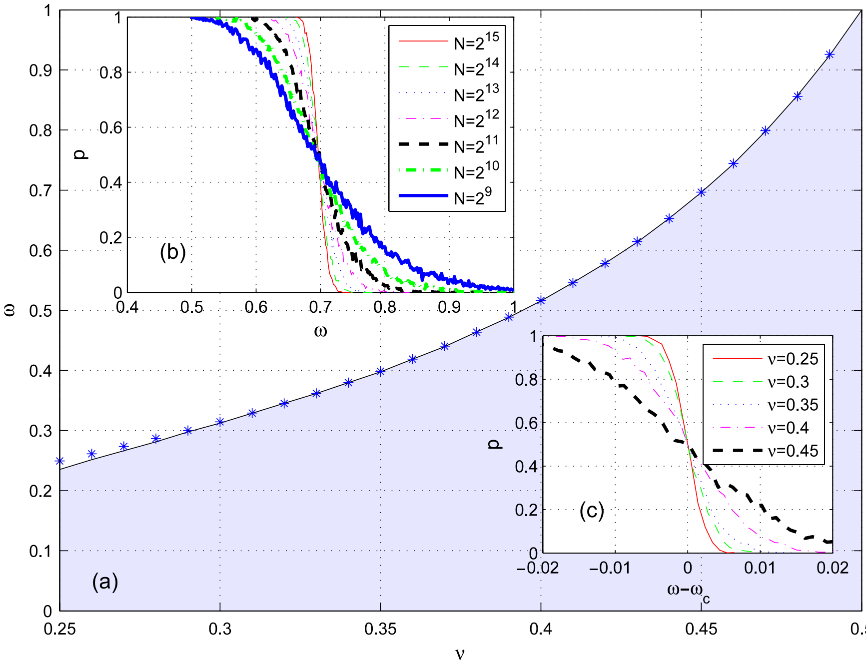

Fig. 2. (a) Bifurcation diagram of ν, ω for $G = 3$. The critical points ω_c are marked as "*" as ν varies. The simulation results are in perfect agreement with the theoretical prediction (solid line). If $\omega < \omega_c$, the initial minority is able to conquer the whole network (shaded area). The network size $N = 2^{15}$. (b) The inset plots the probability p as a function of ω and N with $\nu = 0.45$. p is the percentage of 1000 runs where the opinion of an initial minority dominates the whole network finally. Different symbols correspond to the different network sizes from 2^9 to 2^{15}. The transition sharpens with increasing network size N. (c) The inset depicts the transition with ν varying from 0.25 to 0.45. The transition sharpens with decreasing ν. The probability p is the percentage that the opinion of an initial minority dominates the whole network finally in 1000 runs, and the x-axis is scaled as $\omega - \omega_c$.

and outside of the community inhomogeneous. When $\omega < \omega_c$, this inhomogeneity of connections will make the global minority be the local majority in the group discussion and dominate the whole network finally.

The insets in Fig. 2 show the phase transitions of the evolution. With the growth of network size, the critical value ω_c keeps unchanged. A larger scale network size makes the transition sharper, see Fig. 2(b). However with the fraction of the minority in the network increasing, the transition is smoother in Fig. 2(c).

4. The condition for $G > 3$

The same as the condition for $G = 3$, the dynamics of opinion evolution for the other values of G follow the equations

$$\begin{cases} \dfrac{\mathrm{d}a_\mathrm{i}}{\mathrm{d}t} = \dfrac{1}{\nu}\left[\displaystyle\sum_{k=0}^{G} P_{(G-k,k)} R_{\mathrm{i},(G-k,k)}\right] \\ \dfrac{\mathrm{d}a_\mathrm{o}}{\mathrm{d}t} = \dfrac{1}{1-\nu}\left[\displaystyle\sum_{k=0}^{G} P_{(G-k,k)} R_{\mathrm{o},(G-k,k)}\right] \end{cases} \tag{2}$$

while the calculations of $P_{(G-k,k)}$, $R_{\mathrm{i},(G-k,k)}$ and $R_{\mathrm{o},(G-k,k)}$ for $G > 3$ would be more complex. Take $G = 5$, $k = 2$ for example. In that case, the group is selected with three nodes in the community, two nodes outside with the probability

$$P_{(3,2)} = C_5^3 P_\mathrm{i} P_\mathrm{i} P_\mathrm{i} P_\mathrm{o} P_\mathrm{o} p_1^7 p_2^3 = 10\upsilon^3(1-\upsilon)^2\omega^7 p_2^{10}.$$

All possible conditions for selection (3, 2) are listed in Table 2, with the first three nodes in the community and the last two nodes outside.

The contributions $R_{\mathrm{i},(3,2)}$, $R_{\mathrm{o},(3,2)}$ can be calculated as

$$R_{\mathrm{i},(3,2)} = -3a_\mathrm{i}b_\mathrm{i}^2b_\mathrm{o}^2 - 6a_\mathrm{i}b_\mathrm{i}^2a_\mathrm{o}b_\mathrm{o} + 6a_\mathrm{i}b_\mathrm{i}^2a_\mathrm{o}^2 - 6a_\mathrm{i}^2b_\mathrm{i}b_\mathrm{o}^2 + 6a_\mathrm{i}^2b_\mathrm{i}a_\mathrm{o}b_\mathrm{o} + 3a_\mathrm{i}^2b_\mathrm{i}a_\mathrm{o}^2,$$

$$R_{\mathrm{o},(3,2)} = -2b_\mathrm{i}^3a_\mathrm{o}b_\mathrm{o} - 2b_\mathrm{i}^3a_\mathrm{o}^2 - 6a_\mathrm{i}b_\mathrm{i}^2a_\mathrm{o}b_\mathrm{o} + 6a_\mathrm{i}^2b_\mathrm{i}a_\mathrm{o}b_\mathrm{o} + 2a_\mathrm{i}^3b_\mathrm{o}^2 + 2a_\mathrm{i}^3a_\mathrm{o}b_\mathrm{o}.$$

Like the bifurcation diagram for $G = 3$, both the simulations and theoretical results for $G = 3, 4, 5, 6$ are illustrated in Fig. 3. The critical points ω_c increase with the growth of ν, and the larger group size makes the regions of ν, ω for the minority to win widen markedly. As the group size grows, dominance of the minority becomes easier to achieve in the

Table 2
The possible conditions and their contributions for selection (3, 2).

Case	Probability	Combinations	A_{i}	A_{o}
$(\beta, \beta, \beta \mid \beta, \beta)$	$b_i^3 b_o^2$	$C_3^0 C_2^0 = 1$	0	0
$(\beta, \beta, \beta \mid \beta, \alpha)$	$b_i^3 a_o b_o$	$C_3^0 C_2^1 = 2$	0	−1
$(\beta, \beta, \beta \mid \alpha, \alpha)$	$b_i^3 a_o^2$	$C_3^0 C_2^2 = 1$	0	−2
$(\beta, \beta, \alpha \mid \beta, \beta)$	$a_i b_i^2 b_o^2$	$C_3^1 C_2^0 = 3$	−1	0
$(\beta, \beta, \alpha \mid \beta, \alpha)$	$a_i b_i^2 a_o b_o$	$C_3^1 C_2^1 = 6$	−1	−1
$(\beta, \beta, \alpha \mid \alpha, \alpha)$	$a_i b_i^2 a_o^2$	$C_3^1 C_2^2 = 3$	2	0
$(\beta, \alpha, \alpha \mid \beta, \beta)$	$a_i^2 b_i b_o^2$	$C_3^2 C_2^0 = 3$	−2	0
$(\beta, \alpha, \alpha \mid \beta, \alpha)$	$a_i^2 b_i a_o b_o$	$C_3^2 C_2^1 = 6$	1	1
$(\beta, \alpha, \alpha \mid \alpha, \alpha)$	$a_i^2 b_i a_o^2$	$C_3^2 C_2^2 = 3$	1	0
$(\alpha, \alpha, \alpha \mid \beta, \beta)$	$a_i^3 b_o^2$	$C_3^3 C_2^0 = 1$	0	2
$(\alpha, \alpha, \alpha \mid \beta, \alpha)$	$a_i^3 a_o b_o$	$C_3^3 C_2^1 = 2$	0	1
$(\alpha, \alpha, \alpha \mid \alpha, \alpha)$	$a_i^3 a_o^2$	$C_3^3 C_2^2 = 1$	0	0

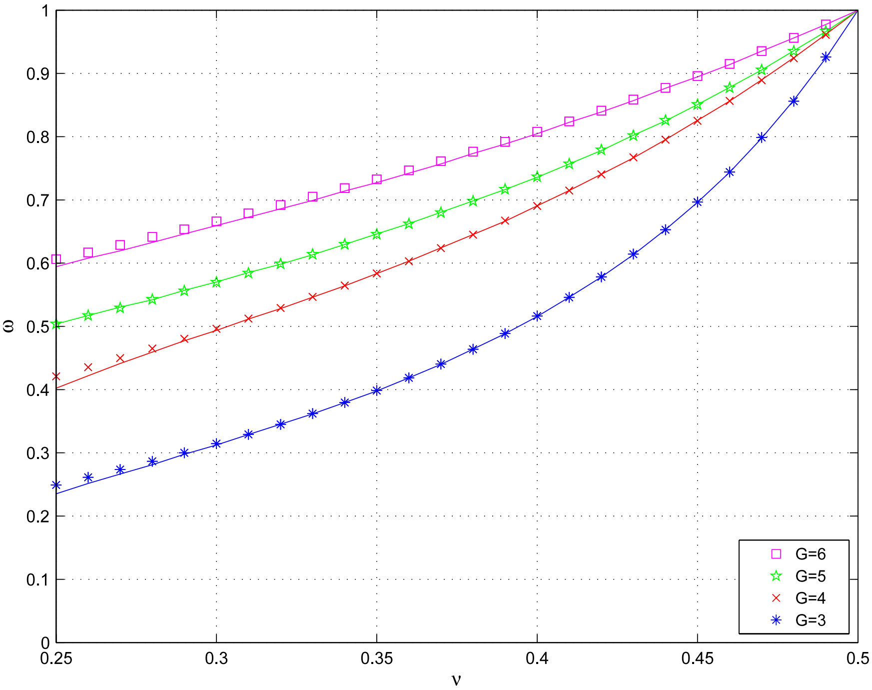

Fig. 3. Bifurcation diagram of ν, ω for $G = 3, 4, 5, 6$, from the bottom up with different symbols correspondingly. The network size $N = 2^{15}$. The simulation results are in perfect agreement with the theoretical prediction (solid line).

group discussion due to the dense internal connection. Hence the community structured minority gains more advantage in larger local scale discussions.

5. Conclusion

In conclusion, to investigate the effect of community structure on the dynamics of opinion spreading, a single community model is proposed in this paper. With both theoretical analysis and computer simulation, we find that a dense connection of the community makes the nodes in it prefer grouping together. If the link probability ratio ω is less than the critical point ω_c, the global minority would be dominant in the local discussion. A larger network size makes the transition sharper. In contrast, with the fraction of the minority in the networks increasing, the transition would be smoother. Moreover, further simulations and computations in different group sizes also indicate that a larger group size may help the community structured minority to achieve success more easily.

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