Physica A 387 (2008) 4665–4672

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

The strength of the minority[☆]

Gan Huang [a], Jinde Cao [a,*], Guanjun Wang [a], Yuzhong Qu [b]

[a] *Department of Mathematics, Southeast University, Nanjing 210096, China*

[b] *Department of Computer Science & Engineering, Southeast University, Nanjing 210096, China*

ARTICLE INFO

Article history:
Received 30 November 2007
Received in revised form 21 February 2008
Available online 3 April 2008

Keywords:
Opinion formation
Community
Cohesion

ABSTRACT

In this paper, we focus on the problem of opinion formation by introducing a simple model with a certain community structure. To understand the strength of the community, we took a particular interest in a special problem of how the opinion of a small but cohesive community could persist or even be finally accepted by the majority of the society. Both simulation and analysis has been done in the absence and presence of noise. In the noiseless environment, assuming the population of the community is fixed, if the cohesion of the community reaches a certain level, then the phase transition will occur in the evolution process that the community will never be assimilated even if it can assimilate the other nodes in the network, which depends on the population of the community. On the other hand, in the presence of noise, the process of opinion formation seems more complex that two transition behaviors occur outside and inside the community as the noise level increases. And the outcomes of the evolution may be completely opposite under different noise conditions.

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1. Introduction

Using simple mathematical models to describe social phenomena in human populations has a long history of study in the social sciences [1–3]. For instance, the classical random graphs theory [4] has been successfully used to analyze all kinds of social networks [5]. Recently, it has been well known that many social and nature networks are quite different from simple random networks and share several unexpected properties such as the small-world property [6], power-law degree distributions [7,8], and network transitivity. It is also reported by Girvan and Newman [9] that community structure could be found in many networks which are composed of many communities of nodes. And the nodes of the same community are highly connected, while there are few links between the nodes of different communities.

It is clear that the evolution of the opinion formation is deeply dependent on the topologies of the networks. In Refs. [10,11], the social impact models are considered under different structures of the networks. In particular, Ref. [12] proposes the majority model in a topology consisting of two coupled fully connected networks to mimic the existence of communities in social networks. Furthermore, an interesting problem that asks whether the dynamics taking place on a network controls the network structure or the structure controls the dynamics is presented in Ref. [13]. Other models such as the KH model by Hegselmann and Krause [14], the voter model [15] and Galam's majority rule [16] are reviewed in Ref. [17]. Most models in these papers always assume that the initial opinions are distributed randomly [13,18]. In conclusion, the initial majority sometimes determines the final state [10,11,18,19], sometimes a perfect equality is obtained or the initial minority wins [20–22]. A general frame for two state opinion models can be seen in Ref. [23].

☆ This work was supported by the 973 Program of China under Grant 2003CB317004.

* Corresponding author. Fax: +86 25 83792316.
E-mail addresses: jdcao@seu.edu.cn, jdcaoseu@gmail.com (J. Cao).

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doi:10.1016/j.physa.2008.03.033

It is obvious that the opinion or taste of the users in the highly cohesive community may strongly differ from the majority, where the cohesion of the community depicts how strong the ties between the members of a social group are and how homogeneous their properties are regarding the overall structure [24]. In an economic context, for instance, it is well known that small cliques of core users can form a niche [25,26] and have a different behavior to the majority. As an example, one may think of the users of Mac computers who concentrate in niche markets like education, graphic arts, and multimedia creation [27]. In Refs. [20–22], Galam etc. focused on the problem of minority opinion under a physical principle called the "inertia principle". This problem highlights one of the fundamental questions in network dynamics, namely, how could the opinion of a small but cohesive community persist or even be accepted by the majority of the society.

To investigate the social impact of the community in the coevolution of the process of opinion formation, a simple model is proposed in this paper, where only one community exists in the network and the population of the community is the minority. It will be found that the strength of the community comes mainly from its cohesion inside the community. It is also influenced by the population of the community and the noise can bring some interesting and unexpected results.

The rest of the paper is organized as follows. In Section 2, the structure of the network and the interactions between opinions are introduced. The dynamic of the evolution under the random graph model is described in Section 3. In Section 4, we present and discuss the simulation results and study the effect of the model parameters under the model with the community structure. Finally, the conclusions are summarized in Section 5.

2. Description of the models

Consider a network with N vertices, numbered from 1 to N and joined in pairs randomly by $\frac{NK}{2}$ edges where K is the average degree of the network. Suppose the first M nodes ($M \leq N/2$) construct a community, hence the community takes a small section of the population of the network. To show the cohesion of the community, we add edges randomly into the subgraph until the average degree of subgraph reaches L. The subgraph is a graph whose vertices and edges form subsets of the vertices and edges of a given graph. In order to make the average degree of subgraph reach L, it can be calculated that $\frac{1}{2}M(L - K\frac{M-1}{N-1})$ edges are needed to be added in average, as the average degree of the subgraph is $K\frac{M-1}{N-1}$ before any edges have been added. Hence, we can construct the topology of the model in the following two steps:

(1) Construct a N-nodes random graph with the average degree to be K.
(2) Add edges into the subgraph composed by the first M nodes.

It is worth noting that, after the two-step construction, the average degree of the nodes outside the community is still K, while the average degree of the nodes inside is neither K nor L, but $K\frac{N-M}{N-1} + L$.

In the network, each of the nodes holds one of the two opposite opinions denoted by $1, -1$. To illustrate the social impact of the community, we always assume the initial distributions of the opinion are as follows:

$$\sigma_i(0) = \begin{cases} 1, & i = 1, \dots, M; \\ -1, & i = M+1, \dots, N \end{cases}$$

where $\sigma_i(t)$ denotes the opinion of the ith node at the time step t. And the dynamics of the opinion are governed by the rule:

$$\sigma_i(t+1) = \operatorname{sign}\left[\sum_{j=1}^{N} d_{ij}\sigma_j(t)\xi_{ji}(t) + h_i(t)\right], \tag{1}$$

applied synchronously to every individual. d_{ij} denotes the connection between the nodes i, j. If the ith node has an edge connecting with the node j then $d_{ij} = 1$, else $d_{ij} = 0$. It is obvious that $d_{ij} = d_{ji}$, and we also assume $d_{ii} = 1$. And the network can be denoted by the matrix $D = (d_{ij})_{n\times n}$. $\xi_{ji}(t)$ is the impact of the node j on node i, which is a stochastic variable of the time t with the uniform distribution on $[0, 1]$. And $h_i(t)$ is the noise of the system which is also a random variable of the time t that distributes uniformly on $[-h, h]$, h is the bound of the noise $h_i(t)$. The rule is developed from the social impact theory [10,28,29], while we suppose that the impact of the individuals are a stochastic variable of the time. In the noiseless environment, i.e. $h = 0$, the nodes tend to accept the opinions of the majority of their neighbors, however they are also influenced by the different impact of their neighbors at the different time.

3. Random graph

In order to compare the evolution process within the model with a community structure, we first present the simulation of the model with the standard random graph, that no edges are added into the graph in the second step of the construction and $L = K\frac{M-1}{N-1}$. Hence M indicates the population of minority of the opinion in the initial distribution. In the following, we fix $N = 1000$ and investigate the evolution of the opinion formation under the different values of K and M in both noisy and noiseless environments.

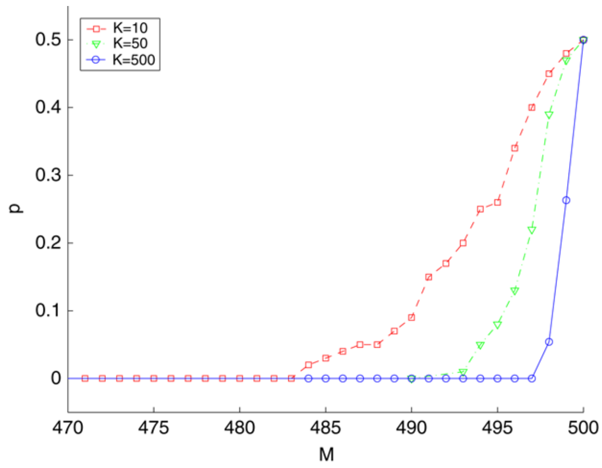

Fig. 1. The evolution of the network without noise (random graph networks). P is the percentage of that the opinion of the community with M members dominate the whole network finally in our 1000 simulations. For the three curves, $K = 10, 50, 500$, respectively. The network size $N = 1000$.

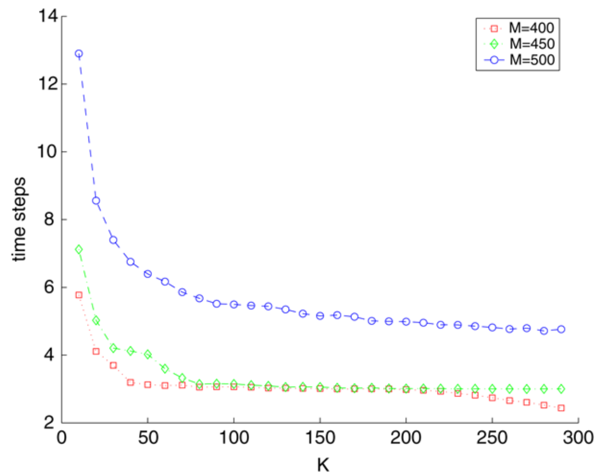

Fig. 2. The time steps to a consensus state in the evolution progress (random graph networks without noise), as the average degrees of the nodes K is increased. $N = 1000$, three curves correspond to $M = 400, 450, 500$, respectively. And each point corresponds to an average over 1000 runs with different random seeds.

3.1. In the absence of noise

As Fig. 1 illustrated in the absence of the noise, if $M = N/2$, the opinion of the M nodes is dominant in the final state with the probability p nearly 50%. However, with the decreasing of M, the probability p drops down sharply. And the rate of descent increases with the growth of K. While if $M \ll N/2$, the final opinion equals that of the initial majority in the most frequencies, i.e. $p = 0$. Computer simulations also indicate that the system converges quickly to the consensus state, that all the nodes have the same opinion, which is show in Fig. 2. It is clear that the step number shows a downturn trend with the increase of K and it declines more quickly at a lower level of K. It seems that, with a closer relationship, the consensus opinion is easier to form.

3.2. In the presence of noise

The effect of noise is shown in Fig. 3. If the noise level is low, the initial minority's opinion will be eliminated ultimately, as the same in the noiseless condition. However, as the noise level grows, the time for the network to hold its opinion is shortened. And a transition occurs between the consensus and uniform distributions, and the transition behavior can be put off with a lager K. For the noise h larger than a certain value, each opinion remains with the same percentages of the population after 5000 steps. And the simulations also indicate that the evolution appears to be less sensitive to the initial distribution of the initial opinions.

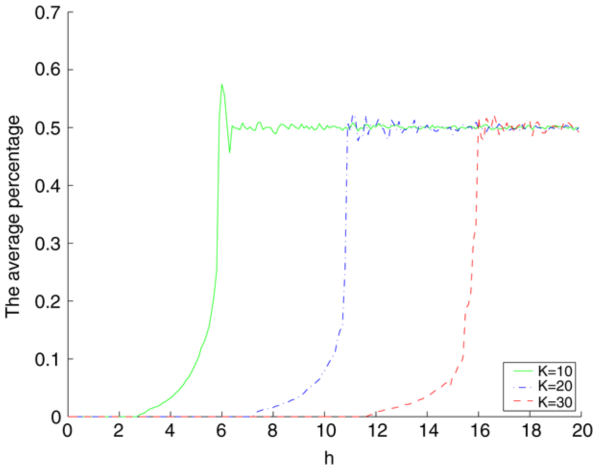

Fig. 3. The average percentage of the nodes which hold the opinion “1” of the initial minority after 5000 steps under different of noise level h with 1000 runs.

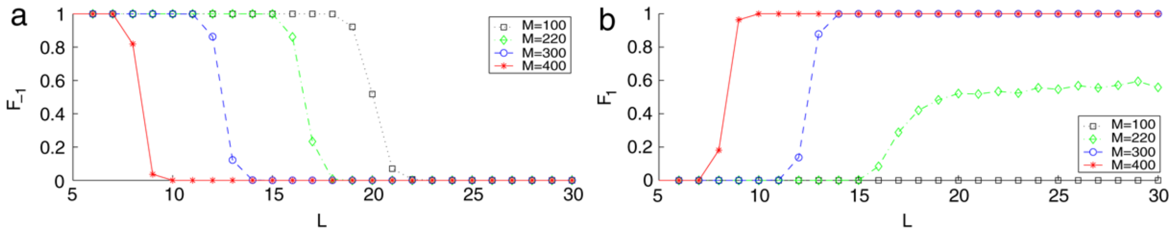

Fig. 4. The evolution results of the noiseless model. Figure (a) shows P^{-1}, the proportion of the final opinion -1 in 1000 runs turn out within 5000 time steps as the value of L is varied. Figure (b) depicts P^1, the proportion of the final opinion 1 in 1000 runs turn out within 5000 time steps as the value of L is varied. Here $N = 1000$, $K = 10$ and the curves in different color represent the evolution result for $M = 100, 220, 300, 400$ correspondingly.

4. Network with community structure

4.1. In the absence of noise

To understand the impact of the community, we first concentrate on the noiseless model, i.e. $h = 0$, and investigate the dynamic behaviors of opinion formation with different values of M, L. To compare with the cohesion of the community, we also set K at a low level as $K = 10$, where the connections between nodes in whole graph are sparse but connected. A connected graph is one where there is a path between any pair of nodes (via zero or more other nodes) in the graph. In the simulation, we run the model 1000 times with each set of M, L and record their evolution results after 5000 steps, which is really a long time period compared with the time consumed in Fig. 2. There are three possible outcomes after 5000 steps as follows: (1) all the nodes have the opinion -1, which equals that of the initial majority; (2) all agree with 1, which is the opinion of the initial minority; (3) the opinions among the individuals haven’t unified.

Naturally, the following two questions must be concerned:

(1) Is it possible that the initial minority’s opinion are able to conquer the whole network? If it is possible, what is the effect of M and L?
(2) In the third condition, what would happen? Could the consensus state be reached finally?

The following simulation will try to answer these questions. Fig. 4 records the results of the evolutions after 5000 steps with increasing L. As is shown in Fig. 4(a), regardless the population of the community M, the final opinion equals that of the initial majority at the low level of L. And if L is larger than a certain value, the opinion -1, which is the opinion of the majority initially, couldn’t be dominant within the first 5000 steps. With L increasing, the transitional activity occurs in the processing of opinion formation in the network. And the transition occurs later when M is larger with $L = 20 \pm 0.5$, 16.5 ± 0.5, 12 ± 0.5, 8.5 ± 0.5 corresponding for $M = 100, 220, 300, 400$. As shown in Fig. 4(b), whereas L is larger than the critical point, the results seem different due to the different value of M. As $M = 300, 400$, the nodes in network hold the unified opinion 1, and the consensus state couldn’t be attached within 5000 steps for $M = 100$. While $M = 220$ seems to be near the transition points that both conditions occur with a certain percent, and the percentage holds on as L increases.

As is noted above, the average degree of the nodes inside the community is $K\frac{N-M}{N-1} + L$, which means the nodes connected with L other nodes inside the community, and $K\frac{N-M}{N-1}$ nodes outside in average. Hence, at the first step of the evolution

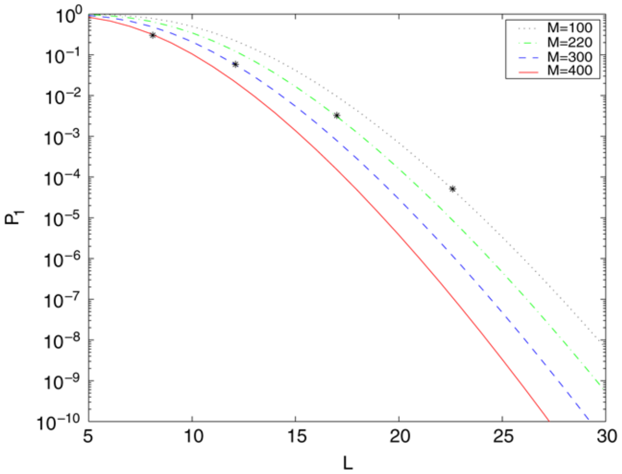

Fig. 5. The value of P_1 varies by L for $M = 100, 220, 300, 400$ in different colors, where P_1 is the probability of that a member of the community changes its opinion in the first step of the evolution process. And the stars, marked on the curves, indicate the value of L where $P_1 = P_2$, which are the transition points illustrated in Fig. 4(a) except for $M = 100$. Here $N = 1000$ and $K = 10$.

progress, the opinion of the nodes inside the community would be changed from 1 to −1 with the probability

$$P_1 = P\left(-\sum_{i=1}^{\frac{K(N-M)}{N-1}} \xi_i + \sum_{j=1}^{L+1} \eta_j < 0\right) = \int_{-\infty}^{0} f_{-\sum_{i=1}^{\frac{K(N-M)}{N-1}} \xi_i + \sum_{j=1}^{L+1} \eta_j}(x)\mathrm{d}x$$

where ξ_i, η_j are stochastic variables with the uniform distribution on [0, 1], and

$$f_{\xi_i}(x) = \begin{cases} 1, & x \in [0, 1] \\ 0, & \text{else} \end{cases}$$

$$f_{\sum_{i=1}^{k} \xi_i}(x) = \int_{-\infty}^{+\infty} f_{\sum_{i=1}^{k-1} \xi_i}(y) f_{\xi_k}(x-y)\mathrm{d}y$$

$$f_{\sum_{i=1}^{n} \xi_{i=1} - \sum_{j=1}^{m} \eta_j}(x) = \int_{-\infty}^{+\infty} f_{\sum_{i=1}^{n} \xi_i}(y) f_{\sum_{j=1}^{m} \eta_j}(y-x)\mathrm{d}y.$$

So the probability would be calculated by computers, which is shown in Fig. 5, with the parameter $K\frac{N-M}{N-1}$ and $L+1$ rounded up. Similarly, the opinion of the nodes outside would been converted with the probability

$$P_2 = P\left(-\sum_{i=1}^{\frac{K(N-M-1)}{N-1}+1} \xi_i + \sum_{j=1}^{\frac{KM}{N-1}} \eta_j > 0\right) = \int_{0}^{\infty} f_{-\sum_{i=1}^{\frac{K(N-M-1)}{N-1}+1} \xi_i + \sum_{j=1}^{\frac{KM}{N-1}} \eta_j}(x)\mathrm{d}x.$$

Obviously, P_2 is independent of L, but depends on N, K, M. The points are also fixed with $P_1 = P_2$ for different M in Fig. 5. As is shown in Fig. 5, the values of L for $P_1 = P_2$ are 22.6, 17, 12, and 8.1 corresponding for $M = 100, 220, 300$, and 400. The critical points coincide with the observed values form Fig. 4(a) except for $M = 100$. In Fig. 5, it could be found that the possibilities of which $P_1 = P_2$ for $M = 220, 300, 400$ are larger that 1.0×10^{-3}, while the possibility for $M = 100$ is nearly 1.0×10^{-4}, so it could hardly occur within 5000 steps. As the increasing of L, the value of P_1 shrinks exponentially. So the community can hold its opinion for an exponentially long time. And these phenomena can also be verified from the simulation.

Hence, in the noiseless environment, a community can display its strength with the highly cohesive structure inside. If the community holds a certain proportion of the population, as $M = 300, 400$ corresponding to $N = 1000$, then the final opinion will be equal to the opinion of the community with the high cohesion. However, a similar phenomenon could hardly be observed in the simulation if M is not large enough, such as $M = 100$ with $N = 1000$, $K = 10$. It is also a theoretically true fact that the opinion of the highly cohesive community will finally dominate the whole world, even if the population of

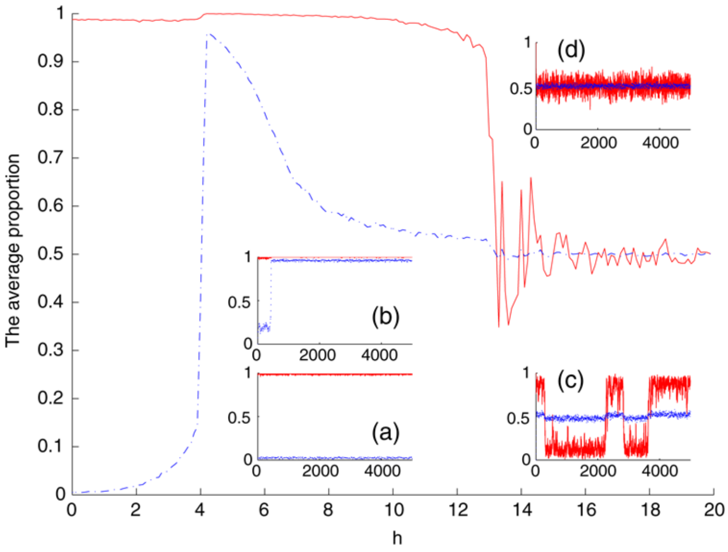

Fig. 6. The evolution of the networks in the presence of different noise where $N = 1000$, $K = 10$, $M = 100$, $L = 25$. h is the largest range of the noise. The blue solid line represents the average proportion of nodes that hold the opinion 1, which is the initial opinion of the community, inside the community, and the red dashed line describes the average proportion outside the community. Each value is an average of 1000 runs with different random seeds. The subfigures (a, b, c, d) depict the typical evolution progress with corresponding value of h as 2, 4.2, 13.5, 20 within the 5000 time steps. The y-axis means the proportion of nodes which hold the opinion 1, and the blue solid and red dashed lines represent the different proportion inside and the outside of the community. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

the community M is not so large. However, as the possibility P_2 is quite tiny, the opinion of the community is difficult to be accepted by the nodes outside and it will take an exponentially long time to finish the evolution which is also meaningless in real problems. Consequently, so-called metastable states will persist in the simulation, where both the nodes inside and outside the community tend to keep their initial opinion, and both two opposite opinions can coexist for a long time.

4.2. In the presence of noise

In this subsection, we will investigate the influence of the noise of the model. Fig. 6 illustrates the average proportion of an initial minority's opinion after the evolution of 5000 steps in both the interior and exterior of the community with $N = 1000$, $K = 10$, $M = 100$, $L = 25$. Two different phase transitions occur inside and outside the community with the growing of the noise level, due to the different ability of anti-noise for the nodes inside and outside the community.

As is shown in Fig. 6(a), with a small noise, the model performs the same evolution with the noiseless condition, where the network appears to be metastable. With increasing noise, the nodes outside the community cannot persist in its opinion for a long time. And the first transition occurs outside the community, that the nodes outside the community holds their opinion for some time, then suddenly change to the opposite one as the influence of the community, which is depicted in Fig. 6(b). It is unhoped-for that, with the noise ranged from 4.1 to 13.1, a larger part of the nodes hold the opinion of the initial minority, which is totally different from the noiseless condition. However, with increasing noise, the proportion goes down quickly outside the community, but it remains the same level inside the community.

With the continuing growth of the noise, another transition occurs inside the community, that the community couldn't persist in it's initial opinion for a long time any more. And the dynamic behaviors perform differently with the different noise after the transition point. In Fig. 6(c) with $h = 13.5$, the nodes inside the community keep their opinion for some time, then suddenly change to the opposite, which in turn persists for another certain time period. While the phenomenon couldn't be observed in Fig. 6(d) at $h = 20$, where the opinions of the nodes appear to be a uniform distribution. It is apparent that the transition behaviors here are more complex than that of the mode under the standard random graph.

To verify if the transitions are influenced by the parameters, we first changed the value of L while keeping the population number of the community $M = 100$. As the value of L increases, the cohesion of the community is growing. And the community has astronger ability to hold its own opinion, meanwhile nothing has changed to the nodes outside. Hence, as is illustrated in Fig. 7(a), if L is large, as $L = 25$, 30, the transition occurring outside the community seems to be the same. However, the community has a stronger ability to keep their initial opinions from the disturbance of noise with a higher level of L which is illustrated in Fig. 7(b). And as L is small, such as $L = 15$ for the simulation in Fig. 7, the transition occurs differently from the others. The initial opinion of the community could not be accepted by the majority of the network at any level of the noise. And the distribution of the opinion inside and outside the community appears similar. It should be noted that some previously unimagined results come out with the moderate value of L. As, for instance, $L = 19$ in the simulation where the majority's opinion can dominate the whole network with the noise range $h \leq 4.1$. However, if $h \in [4.2, 10]$, then

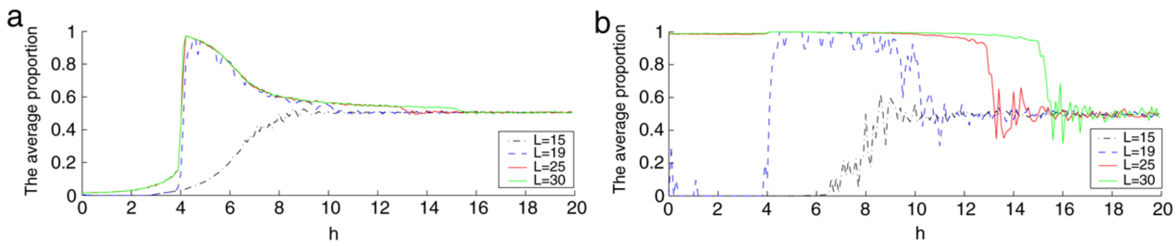

Fig. 7. The average proportion of nodes holding the opinion of the initial minority after 5000 time steps, as the value of the noise range h is varied for $N = 1000$, $K = 10$, $M = 100$. Figure (a) and (b) depict the proportions inside and outside the community correspondingly. And the curves represent the different proportions under different values of L.

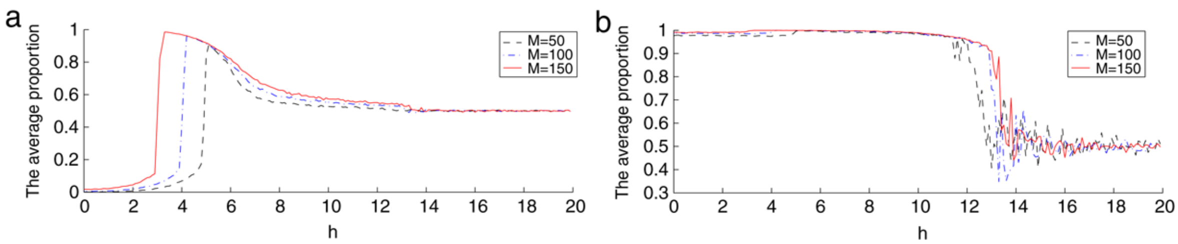

Fig. 8. The average proportion of nodes holding the opinion of the initial minority after 5000 time steps, as the value of the noise range h is varied. Figure (a) and (b) depict the proportions inside and outside the community correspondingly. Here $N = 1000$, $K = 10$, and L is fixed as $L = 25$, but M is varied from 50 to 150.

the converse happens. In particular, when the noise range $h = 4.5$, more than 95% of the nodes, both inside and outside the community, hold the initial minority's opinion 1.

Now we turn our attention to the influence of the parameters M. Different from the influence of L, the second transition evidently seems not be affected by population of the community M in Fig. 8(b). While in Fig. 8(a), the effect of increasing the value of M translates itself as an leftward shift of the curve, and the peak of the curve is shifted upward. As an unexpected result that the opinion of the initial minority can dominate the majority in the network at a certain level of the noise, even if the population is really small.

5. Conclusion

In this paper, we have investigated the social impact of the community in the evolutionary process of opinion formation, where the community is the minority in the network. Both the noisy and the noiseless condition is considered.

In the absence of noise, the strength of the community is determined by its cohesion. However, a highly cohesive community can show its strength in different ways according to the size of the population. If the population of the community reaches a certain proportion of the network, the opinion of a highly cohesive community can dominate the whole world. Furthermore, if the population is not large enough, the community also has its ability to hold its original opinion for an exponentially long time.

In the presence of noise, different transition behaviors occur inside and outside the community at the different time. It is interesting and unexpected that if the noise level lies between the two transition points, the community can dominate the network with high cohesion inside. Hence, in consideration of the novel feature we found, the community can also take measures to adjust the noise to make its opinion be accepted by the majority of the society, even if the population of the community is really small.

And some other interesting problems we will also be concerned ourselves with in further research, such as how to limit or reinforce the strength of the community or the competition between different type of communities.

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