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Multistability in bidirectional associative memory neural networks ☆

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Abstract

In this Letter, the multistability issue is studied for Bidirectional Associative Memory (BAM) neural networks. Based on the existence and stability analysis of the neural networks with or without delay, it is found that the $2n$-dimensional networks can have 3^n equilibria and 2^n equilibria of them are locally exponentially stable, where each layer of the BAM network has n neurons. Furthermore, the results has been extended to $(n+m)$-dimensional BAM neural networks, where there are n and m neurons on the two layers respectively. Finally, two numerical examples are presented to illustrate the validity of our results.

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Keywords: Multistability; BAM neural networks; Delay; Equilibrium

1. Introduction

In recent years, neural networks have attracted more and more attention of researchers due to their great perspectives of application. Ranging from classifications, associative memory, image processing, and pattern recognition to parallel computation and its ability to solve optimization problems, neural networks work as an intelligent tool in different situations. Neural networks have complex dynamical behaviors, such as stability [1–5], periodic bifurcation and chaos [6–9], which have been extensively investigated. The theory on the dynamics of the networks have been developed according to the purposes of applications.

In the applications of neural networks for associative memory storage or pattern recognition, the coexistence of multiple equilibria is a necessary feature [10–13]. The notion of "multistability" of a neural network is used to describe coexistence of multiple stable patterns. In [14], the multistability of the delayed neural networks was discussed:

$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j\big(x_j(t-\tau_{ij})\big) + J_i, \quad i = 1, 2, \ldots, n. \tag{1}$$

It is found that an n-neuron cellular neural networks can have up to 2^n locally stable equilibria. Ref. [15] studied a general delayed neural networks:

$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j\big(x_j(t)\big) + \sum_{j=1}^{n} \beta_{ij} g_j\big(x_j(t-\tau_{ij})\big) + J_i, \quad i = 1, 2, \ldots, n. \tag{2}$$

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In Refs. [16–18], the multistability of cellular neural networks (CNNs) and delayed cellular neural networks (DCNNs) was investigated. Furthermore, in [15,17], the authors studied multiperiodicity and exponential attractivity of neural networks evoked by periodic external inputs. In [25], the multistability and multiperiodicity are discussed for a class of delayed Cohen–Grossberg neural networks. While many multistability criterions depend deeply on the self-connection weights α_{ii}, β_{ii} in the neural networks. For instance in [14–16], the related criterions always require α_{ii} or $\alpha_{ii}+\beta_{ii}$ to be positive. However, if $\alpha_{ii}=0$, $\beta_{ii}=0$, the criterions mentioned above are not applicable for checking the multistability of the neural networks.

Bidirectional Associative Memory (BAM) network, introduced by Kosko in [19–21], is a typical neural network model, in which the self-connections of all neurons are zero. It has been successfully applied to pattern recognition and associative memory. As an extension of the unidirectional autoassociator of Hopfield neural networks, BAM neural network is formed by neurons arranged in two layers. The neurons in one layer are fully interconnected to the neurons in the other layer, while there are no interconnections among neurons in the same layer. In Refs. [6,8,22,23], the authors discussed the problem of stability and periodic for BAM networks with or without axonal signal transmission delays. However, to the best of our knowledge, few papers (if any) are concerned with the multistability of BAM neural networks.

Motivated by the above discussions, we shall study the multistability of BAM neural networks in this letter. In Sections 2 and 3, the $2n$-dimensional networks are considered with n-neurons on each layer of the BAM networks. The condition of the existence of multiple equilibria is obtained, in Section 2. In Section 3, the stability of the equilibria is investigated with delay or without delay. In Section 4, the neural network model is extended to a more general form, in which there can be different number of neurons on the two layers. Both the global stability and local metastability conclusions are obtained. In Section 5, two illustrative examples are provided with simulation results. Finally, conclusions are given in Section 6.

2. Existence of multiple equilibria

In this section and Section 3, we consider the BAM neural networks without delay or with delay, respectively as follows:

$$\begin{cases} \dot{x}_i(t) = -a_i x_i(t) + \sum_{1\leqslant j\leqslant n} b_{ij} g(y_j(t)) + I_i, \\ \dot{y}_i(t) = -c_i y_i(t) + \sum_{1\leqslant j\leqslant n} d_{ij} g(x_j(t)) + J_i, \end{cases} \quad i = 1,2,\dots,n \tag{3}$$

$$\begin{cases} \dot{x}_i(t) = -a_i x_i(t) + \sum_{1\leqslant j\leqslant n} b_{ij} g(y_j(t-\tau_{ij})) + I_i, \\ \dot{y}_i(t) = -c_i y_i(t) + \sum_{1\leqslant j\leqslant n} d_{ij} g(x_j(t-\sigma_{ij})) + J_i, \end{cases} \quad i = 1,2,\dots,n \tag{4}$$

where $a_i > 0$, $c_i > 0$, x_i, y_j are the activations of the ith neurons and jth neurons in the two layers, respectively. b_{ij}, d_{ij} are the connection weights through the neurons in two layers, and I_i and J_i denote the external inputs. $\tau_{ij} > 0$, $\sigma_{ij} > 0$ correspond to finite speed of axonal signal transmission. Denote $\tau := \max_{1<i,j<n}\{\tau_{ij}, \sigma_{ij}\}$, where $\tau > 0$. The activation function $g(s) = \tanh(s)$, which holds the sigmoidal configuration and is nondecreasing with saturation. As a functional differential equations described by system (4), the initial condition is

$$\begin{cases} x_i(\theta) = \phi_i(\theta), \\ y_i(\theta) = \psi_i(\theta), \end{cases} \quad \theta \in [-\tau, 0],$$

where $\phi_i, \psi_i \in \mathcal{C}([-\tau,0],\mathbb{R})$.

The stationary equations of systems (3) and (4) are identical as follows,

$$\begin{cases} -a_i x_i + \sum_{1\leqslant j\leqslant n} b_{ij} g(y_j) + I_i = 0, \\ -c_i y_i + \sum_{1\leqslant j\leqslant n} d_{ij} g(x_j) + J_i = 0, \end{cases} \quad i = 1,2,\dots,n. \tag{5}$$

Firstly, consider the BAM neural network with a single couple of neurons,

$$\begin{cases} \dot{x}(t) = -a_i x(t) + b_{ii} g(y(t)) + I_i, \\ \dot{y}(t) = -c_i y(t) + d_{ii} g(x(t)) + J_i. \end{cases} \tag{6}$$

Hence, the stationary equations can be rewritten as

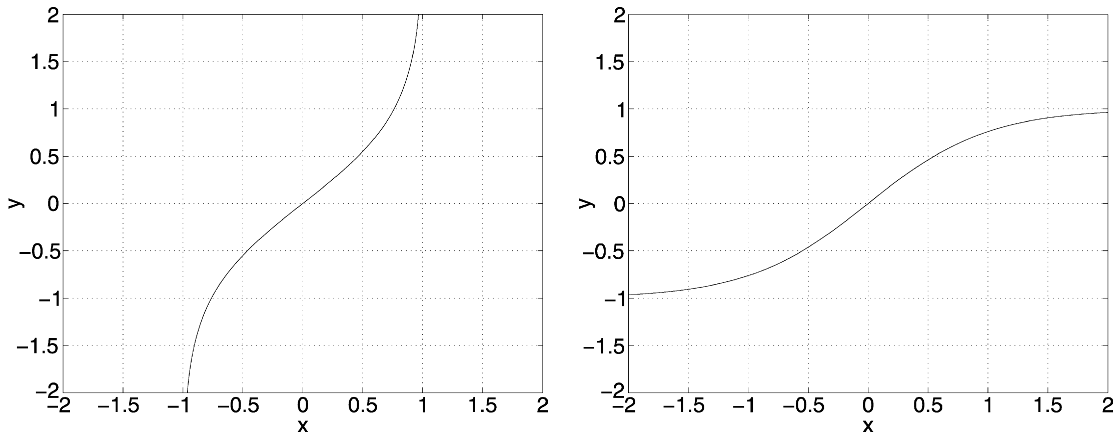
$$\begin{cases} x = \frac{b_{ii}}{a_i} g(y) + \frac{I_i}{a_i} := G_i(y), \\ y = \frac{d_{ii}}{c_i} g(x) + \frac{J_i}{c_i} := H_i(x). \end{cases} \tag{7}$$

As is shown in Fig. 1, the equilibria of Eq. (6) are the crossing points of the curves $x = G_i(y)$ and $y = H_i(x)$.

Here, we propose the first condition:

$$(\mathrm{H}_1):\quad \frac{b_{ii} d_{ii}}{a_i c_i} > 1, \quad i = 1,2,\dots,n.$$

It's worth noting that, as $a_i, c_i > 0$, condition (H_1) also implies $b_{ii} d_{ii} > 0$ for all $i = 1,2,\dots,n$.



(a) (b)

Fig. 1. The graph of $x = G_i(y)$ and $y = H_i(x)$ with $a_i = c_i = b_{ii} = d_{ii} = 1$ and $I_i = J_i = 0$. (a) $x = G_i(y)$. (b) $y = H_i(x)$.

Theorem 2.1. *For $\frac{b_{ii}d_{ii}}{a_i c_i} \leqslant 1$ and $b_{ii}d_{ii} > 0$, system (6) has one unique equilibrium.*

Proof. (1) If $b_{ii} > 0$, then $d_{ii} > 0$, $a_i c_i - b_{ii}d_{ii} > 0$. Hence $G(y)$, $H(x)$ is monotonous increase, and

$$\begin{aligned}
&\lim_{y\to-\infty} G_i(y) = \frac{-b_{ii}+I_i}{a_i}, &&\lim_{x\to-\infty} H_i(x) = \frac{-d_{ii}+J_i}{c_i},\\
&\lim_{y\to+\infty} G_i(y) = \frac{b_{ii}+I_i}{a_i}, &&\lim_{x\to+\infty} H_i(x) = \frac{d_{ii}+J_i}{c_i},\\
&\frac{dG_i}{dy} \in \left(0, \frac{b_{ii}}{a_i}\right], &&\frac{dH_i}{dx} \in \left(0, \frac{d_{ii}}{c_i}\right].
\end{aligned}$$

Therefore the inverse function $G^{-1}(x)$ can be defined on the interval $(\frac{-b_{ii}+I_i}{a_i}, \frac{b_{ii}+I_i}{a_i})$, $\frac{dG_i^{-1}}{dx} > \frac{a_i}{b_{ii}}$ and

$$\begin{aligned}
&\lim_{x\to\frac{-b_{ii}+I_i}{a_i}} G_i^{-1}(x) = -\infty, &&\lim_{x\to\frac{b_{ii}+I_i}{a_i}} G_i^{-1}(x) = +\infty,\\
&\lim_{x\to\frac{-b_{ii}+I_i}{a_i}} \bigl(G_i^{-1}(x) - H_i(x)\bigr) = -\infty, &&\lim_{x\to\frac{b_{ii}+I_i}{a_i}} \bigl(G_i^{-1}(x) - H_i(x)\bigr) = +\infty.
\end{aligned}$$

By means of intermediate value theorem, there exists a x_0 such that $G_i^{-1}(x_0) - H_i(x_0) = 0$. Since

$$\frac{d(G_i^{-1} - H_i)}{dx} > \frac{a_i}{b_{ii}} - \frac{d_{ii}}{c_i} = \frac{a_i c_i - b_{ii}d_{ii}}{b_{ii}c_i} > 0,$$

the value of $G_i^{-1}(x) - H_i(x)$ monotonously increases on the interval $(\frac{-b_{ii}+I_i}{a_i}, \frac{b_{ii}+I_i}{a_i})$, the zero point for $G_i^{-1}(x) - H_i(x)$ is unique. So $(x_0, H_i(x_0))$ is the unique equilibrium for system (6).

(2) If $b_{ii} < 0$, then $d_{ii} < 0$, $a_i c_i - b_{ii}d_{ii} > 0$. With the similar analysis, we have

$$\lim_{x\to\frac{b_{ii}+I_i}{a_i}} \bigl(G_i^{-1}(x) - H_i(x)\bigr) = +\infty, \qquad \lim_{x\to\frac{-b_{ii}+I_i}{a_i}} \bigl(G_i^{-1}(x) - H_i(x)\bigr) = -\infty;$$

and

$$\frac{d(G_i^{-1} - H_i)}{dx} < \frac{a_i}{b_{ii}} - \frac{d_{ii}}{c_i} = \frac{a_i c_i - b_{ii}d_{ii}}{b_{ii}c_i} < 0,$$

$G_i^{-1}(x) - H_i(x)$ monotonously decreases on the interval $(\frac{b_{ii}+I_i}{a_i}, \frac{-b_{ii}+I_i}{a_i})$. So $(x_0, H_i(x_0))$ is also the unique equilibrium for system (6). This completes the proof. □

Remark 1. If $b_{ii}d_{ii} < 0$, the monotonicity of $G_i^{-1}(x) - H_i(x)$ is obviously. Hence it is easy to prove that the equilibrium for system (6) is unique. From Theorem 2.1, we can see that condition (H_1) is a necessary condition for system (6) to have multiple equilibrium.

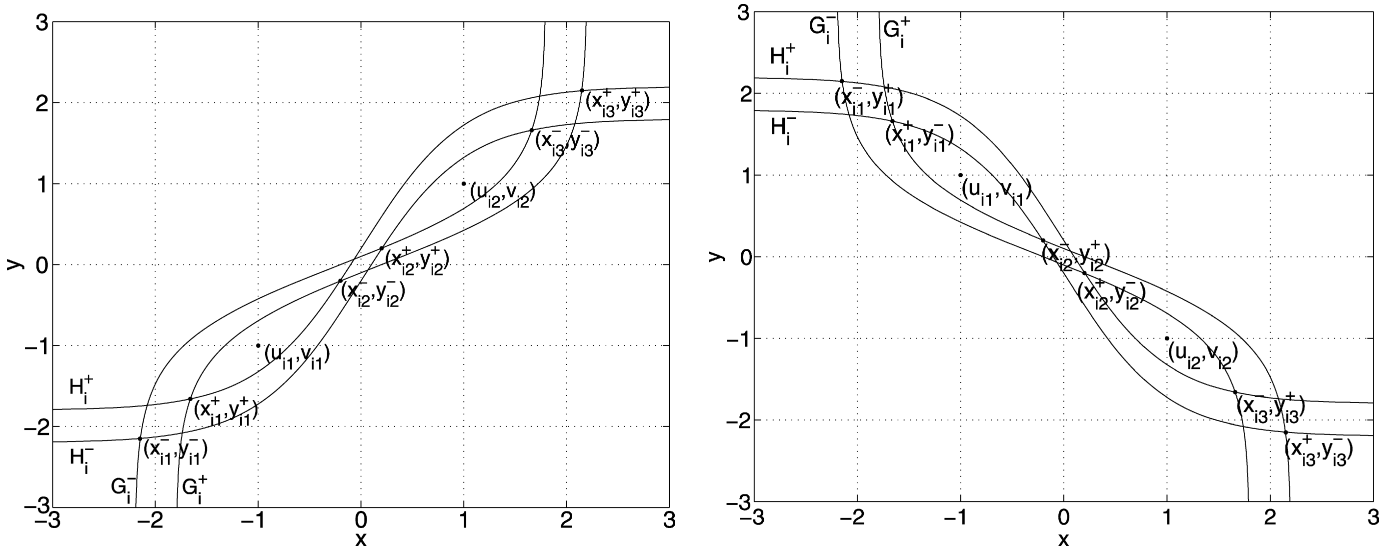


Fig. 2. The configurations of $G_i^+(y)$, $G_i^-(y)$ and $H_i^+(x)$, $H_i^-(x)$. (a) $b_{ii}, d_{ii} > 0$. (b) $b_{ii}, d_{ii} < 0$.

Next, some notations are employed for the convenience of proof.

$$G_i^+(y_i) = \frac{1}{a_i}\left(b_{ii}g(y_i) + I_i + \sum_{j=1, j\neq i}^{n} |b_{ij}|\right), \qquad G_i^-(y_i) = \frac{1}{a_i}\left(b_{ii}g(y_i) + I_i - \sum_{j=1, j\neq i}^{n} |b_{ij}|\right),$$

$$H_i^+(x_i) = \frac{1}{c_i}\left(d_{ii}g(x_i) + J_i + \sum_{j=1, j\neq i}^{n} |d_{ij}|\right), \qquad H_i^-(x_i) = \frac{1}{c_i}\left(d_{ii}g(x_i) + J_i - \sum_{j=1, j\neq i}^{n} |d_{ij}|\right),$$

for $i = 1, 2, \ldots, n$. It is easy to verify that

$$G_i^-(y_i) \leqslant \frac{1}{a_i}\left(\sum_{1\leqslant j\leqslant n} b_{ij}g(y_j) + I_i\right) \leqslant G_i^+(y_i);$$

$$H_i^-(x_i) \leqslant \frac{1}{c_i}\left(\sum_{1\leqslant j\leqslant n} d_{ij}g(x_j) + J_i\right) \leqslant H_i^+(x_i). \tag{8}$$

We make the second assumption which is concerned with the existence of multiple equilibria for systems (3) and (4):
(H_2): for $i = 1, 2, \ldots, n$, there exist two points (u_{i1}, v_{i1}) and (u_{i2}, v_{i2}), where $u_{i1} < u_{i_2}$, such that

$$\begin{cases} \text{if } b_{ii}, d_{ii} > 0, & H_i^+(u_{i1}) < v_{i1}, G_i^+(v_{i1}) < u_{i1}, \\ & H_i^-(u_{i2}) > v_{i2}, G_i^-(v_{i2}) > u_{i2}; \\ \text{if } b_{ii}, d_{ii} < 0, & H_i^-(u_{i1}) > v_{i1}, G_i^+(v_{i1}) < u_{i1}, \\ & H_i^+(u_{i2}) < v_{i2}, G_i^-(v_{i2}) > u_{i2}. \end{cases}$$

The configuration that motivates (H_2) is depicted in Fig. 2. Under assumptions (H_1) and (H_2), if $b_{ii}, d_{ii} > 0$ in Fig. 2(a), then there exist three couples of points (x_{i1}^+, y_{i1}^+), (x_{i1}^-, y_{i1}^-), (x_{i2}^+, y_{i2}^+), (x_{i2}^-, y_{i2}^-), (x_{i3}^+, y_{i3}^+), (x_{i3}^-, y_{i3}^-), where (x_{i1}^+, y_{i1}^+), (x_{i2}^-, y_{i2}^-), (x_{i3}^+, y_{i3}^+) are the crossing points of the curves $y_i = H_i^+(x_i)$, $x_i = G_i^+(y_i)$, (x_{i1}^-, y_{i1}^-), (x_{i2}^+, y_{i2}^+), (x_{i3}^-, y_{i3}^-) are the crossing points of the curves $y_i = H_i^-(x_i)$, $x_i = G_i^-(y_i)$. $x_{i1}^- < x_{i1}^+ < x_{i2}^- < x_{i2}^+ < x_{i3}^- < x_{i3}^+$, $y_{i1}^- < y_{i1}^+ < y_{i2}^- < y_{i2}^+ < y_{i3}^- < y_{i3}^+$.

For $b_{ii}, d_{ii} < 0$ in Fig. 2(b), there exist three couples of points (x_{i1}^-, y_{i1}^+), (x_{i1}^+, y_{i1}^-), (x_{i2}^-, y_{i2}^+), (x_{i2}^+, y_{i2}^-), (x_{i3}^-, y_{i3}^+), (x_{i3}^+, y_{i3}^-), where (x_{i1}^-, y_{i1}^+), (x_{i2}^+, y_{i2}^-), (x_{i3}^-, y_{i3}^+) are the crossing points of the curves $y_i = H_i^+(x_i)$, $x_i = G_i^-(y_i)$, (x_{i1}^+, y_{i1}^-), (x_{i2}^-, y_{i2}^+), (x_{i3}^+, y_{i3}^-) are the crossing points of the curves $y_i = H_i^-(x_i)$, $x_i = G_i^+(y_i)$. $x_{i1}^- < x_{i1}^+ < x_{i2}^- < x_{i2}^+ < x_{i3}^- < x_{i3}^+$, $y_{i3}^- < y_{i3}^+ < y_{i2}^- < y_{i2}^+ < y_{i1}^- < y_{i1}^+$.

Remark 2. Condition (H_2) can be simplified as follows:
(H_2'): for $i = 1, 2, \ldots, n$, there exist u_{i1}, u_{i2}, where $u_{i1} < u_{i2}$, such that

$$\begin{cases} \text{if } b_{ii}, d_{ii} > 0, & G_i^+(H_i^+(u_{i1})) < u_{i1}, \quad G_i^-(H_i^-(u_{i2})) > u_{i2}, \\ \text{if } b_{ii}, d_{ii} < 0, & G_i^+(H_i^-(u_{i1})) < u_{i1}, \quad G_i^-(H_i^+(u_{i2})) > u_{i2}. \end{cases}$$

For $b_{ii}, d_{ii} > 0$, $G_i^+(y)$, $G_i^-(y)$ and $H_i^+(x)$, $H_i^-(x)$ are monotone increasing functions. Due to $H_i^+(u_{i1}) < v_{i1}$, $G_i^+(v_{i1}) < u_{i1}$, we have $G_i^+(H_i^+(u_{i1})) < u_{i1}$, and $H_i^+(G_i^+(v_{i1})) < v_{i1}$. On the other hand, suppose $b_{ii}, d_{ii} > 0$, and $G_i^+(H_i^+(u_{i1})) < u_{i1}$. Let $v_{i1} = (H_i^+(u_{i1}) + [G_i^+]^{-1}(u_{i1}))/2$, then (u_{i1}, v_{i1}) satisfies the inequality $H_i^+(u_{i1}) < v_{i1}$, $G_i^+(v_{i1}) < u_{i1}$. Similarly, if $b_{ii}, d_{ii} > 0$, then $H_i^-(u_{i2}) > v_{i2}$, $G_i^-(v_{i2}) > u_{i2}$ are equivalent to $G_i^-(H_i^-(u_{i2})) > u_{i2}$. Furthermore, the analysis for $b_{ii}, d_{ii} < 0$ is similar. Hence, (H_2') is equivalent to (H_2).

In the following, the existence of equilibria will be proved.

Theorem 2.2. *Under assumptions* (H_1) *and* (H_2), *both systems* (3) *and* (4) *have* 3^n *equilibria.*

Proof. The equilibria of systems (3) and (4) are the roots of Eqs. (5). Under conditions (H_1) and (H_2), the graphs of $x_i = G_i^+(y_i)$, $x_i = G_i^-(y_i)$, $y_i = H_i^+(x_i)$, $y_i = H_i^-(x_i)$ defined above are depicted in Fig. 2. According to the configurations, there are 3^n disjoint closed regions in $\mathbb{R}^{2n}$. Set $\Omega^\alpha = \{(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \in \mathbb{R}^{2n} \mid (x_i, y_i) \in \Omega_i^{\alpha_i}\}$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and $\alpha_i = 1, 2, 3$, where

$$\begin{aligned} \Omega_i^1 &:= \{(x, y) \in \mathbb{R}^2 \mid (x, y) \in [x_{i1}^-, x_{i1}^+] \times [y_{i1}^-, y_{i1}^+]\}, \\ \Omega_i^2 &:= \{(x, y) \in \mathbb{R}^2 \mid (x, y) \in [x_{i2}^-, x_{i2}^+] \times [y_{i2}^-, y_{i2}^+]\}, \\ \Omega_i^3 &:= \{(x, y) \in \mathbb{R}^2 \mid (x, y) \in [x_{i3}^-, x_{i3}^+] \times [y_{i3}^-, y_{i3}^+]\}. \end{aligned} \tag{9}$$

Consider any fixed one of these regions Ω^α. For a given $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n) \in \Omega^\alpha$, we solve

$$-a_i x_i(t) + b_{ii} g(y_i(t)) + \sum_{j=1, j\neq i}^{n} b_{ij} g(\tilde{y}_j(t)) + I_i = 0, \tag{10}$$

$$-c_i y_i(t) + d_{ii} g(x_i(t)) + \sum_{j=1, j\neq i}^{n} d_{ij} g(\tilde{x}_j(t)) + J_i = 0, \tag{11}$$

for x_i, y_i, $i = 1, 2, \ldots, n$. According to an estimate similar to (8), the graph of Eq. (10) lies in the regions between $x_i = G_i^+(y_i)$ and $x_i = G_i^-(y_i)$, while Eq. (11) lies in the regions between $y_i = H_i^+(x_i)$ and $y_i = H_i^-(x_i)$. Thus, there exist at least three solutions, and each of them lies in one of regions in (9) for each i. Consider the one lying in $\Omega_i^{\alpha_i}$ and set it as $(\bar{x}_i, \bar{y}_i)$ for each i, and define a mapping $F_\alpha : \Omega^\alpha \to \Omega^\alpha$ by $F_\alpha(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = (\bar{\mathbf{x}}, \bar{\mathbf{y}}) = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)$. Since g is continuous, the map F_α is continuous. From Brouwer's fixed point theorem, there exists one fixed point $(\mathbf{x}, \mathbf{y})$ of F_α, which is also a zero of Eq. (5). Hence, there exist 3^n equilibria for system Eqs. (3) and (4), and each of them lies in one of the 3^n regions Ω^α. The proof is completed. □

3. Stability analysis

In this section, the stability of the equilibria is considered. The third criterion is proposed concerning stability:

$$(H_3): \quad \begin{cases} -a_i + \sum_{j=1}^n |b_{ij}| g'(\eta_j) < 0, & g'(\eta_j) := \max\{g'(y_j) \mid y_j \in [y_{j1}^-, y_{j1}^+] \cup [y_{j3}^-, y_{j3}^+]\}, \\ -c_i + \sum_{j=1}^n |d_{ij}| g'(\xi_j) < 0, & g'(\xi_j) := \max\{g'(x_j) \mid x_j \in [x_{j1}^-, x_{j1}^+] \cup [x_{j3}^-, x_{j3}^+]\}. \end{cases}$$

From condition (H_2), if $b_{ii}, d_{ii} > 0$, then there exist (u_{i1}, v_{i1}) and (u_{i2}, v_{i2}), where $u_{i1} < u_{i2}$, that $H_i^+(u_{i1}) < v_{i1}$, $G_i^+(v_{i1}) < u_{i1}$. There exists two open region

$$\begin{aligned} D_{i1}^+ &:= \{(x, y) \mid H_i^+(x) < y, G_i^+(y) < x, x_{i1}^+ < x < x_{i2}^-\}, \\ D_{i1}^- &:= \{(x, y) \mid H_i^-(x) > y, G_i^-(y) > x, x < x_{i1}^-\}. \end{aligned}$$

Hence, as is shown in Fig. 3(a), $(u_{i1}, v_{i1}) \in D_{i1}^+$ and (x_{i1}^+, y_{i1}^+) on the edge of D_{i1}^+, (x_{i1}^-, y_{i1}^-) on the edge of D_{i1}^-.

As illustrated in Fig. 3(b), the similar region D_{i3}^-, D_{i3}^+ can be defined as follows

$$\begin{aligned} D_{i3}^- &:= \{(x, y) \mid H_i^-(x) > y, G_i^-(y) > x, x_{i2}^+ < x < x_{i3}^-\}, \\ D_{i3}^+ &:= \{(x, y) \mid H_i^+(x) < y, G_i^+(y) < x, x > x_{i3}^+\}, \end{aligned}$$

which are open and $(u_{i2}, v_{i2}) \in D_{i3}^-$.

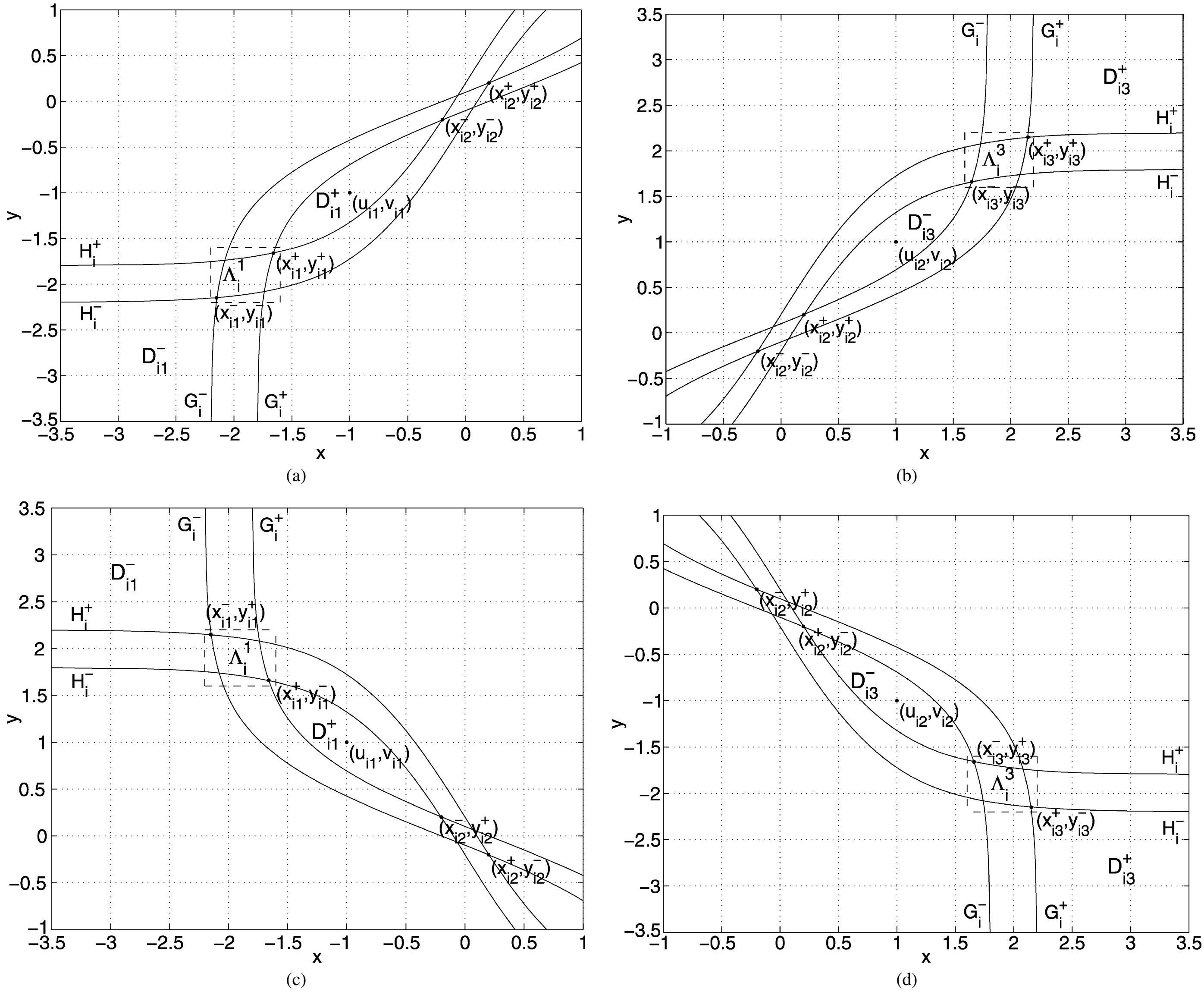


Fig. 3. The regions of D_{i1}^-, D_{i1}^+, D_{i3}^-, D_{i3}^+ and Λ_i^1, Λ_i^3 for $b_{ii}, d_{ii} > 0$ in (a), (b), and $b_{ii}, d_{ii} < 0$ in (c), (d). (a) The regions of D_{i1}^-, D_{i1}^+ and Λ_i^1 for $b_{ii}, d_{ii} > 0$. (b) The regions of D_{i3}^-, D_{i3}^+ and Λ_i^3 for $b_{ii}, d_{ii} > 0$. (c) The regions of D_{i1}^-, D_{i1}^+ and Λ_i^1 for $b_{ii}, d_{ii} < 0$. (d) The regions of D_{i3}^-, D_{i3}^+ and Λ_i^3 for $b_{ii}, d_{ii} < 0$.

According to condition (H_3) and the continuity of the activation function g, there exists a positive constant ϵ_0 and four points $(\tilde{x}_{i1}^-, \tilde{y}_{i1}^-)$, $(\tilde{x}_{i1}^+, \tilde{y}_{i1}^+)$, $(\tilde{x}_{i3}^-, \tilde{y}_{i3}^-)$, $(\tilde{x}_{i3}^+, \tilde{y}_{i3}^+)$, where

$$\begin{aligned}
&(\tilde{x}_{i1}^-, \tilde{y}_{i1}^-) \in \mathcal{B}\big((x_{i1}^-, y_{i1}^-), \epsilon_0\big) \cap D_{i1}^-, &&H_i^-(\tilde{x}_{i1}^-) > \tilde{y}_{i1}^-, G_i^-(\tilde{y}_{i1}^-) > \tilde{x}_{i1}^-,\\
&(\tilde{x}_{i1}^+, \tilde{y}_{i1}^+) \in \mathcal{B}\big((x_{i1}^+, y_{i1}^+), \epsilon_0\big) \cap D_{i1}^+, &&H_i^+(\tilde{x}_{i1}^+) < \tilde{y}_{i1}^+, G_i^+(\tilde{y}_{i1}^+) < \tilde{x}_{i1}^+,\\
&(\tilde{x}_{i3}^-, \tilde{y}_{i3}^-) \in \mathcal{B}\big((x_{i3}^-, y_{i3}^-), \epsilon_0\big) \cap D_{i3}^-, &&H_i^-(\tilde{x}_{i3}^-) > \tilde{y}_{i3}^-, G_i^-(\tilde{y}_{i3}^-) > \tilde{x}_{i3}^-,\\
&(\tilde{x}_{i3}^+, \tilde{y}_{i3}^+) \in \mathcal{B}\big((x_{i3}^+, y_{i3}^+), \epsilon_0\big) \cap D_{i3}^+, &&H_i^+(\tilde{x}_{i3}^+) < \tilde{y}_{i3}^+, G_i^+(\tilde{y}_{i3}^+) < \tilde{x}_{i3}^+,
\end{aligned}$$

such that

$$\begin{cases} a_i > \sum_{j=1}^n |b_{ij}| g'(\eta_j), \\ c_i > \sum_{j=1}^n |d_{ij}| g'(\xi_j), \end{cases} \quad i = 1, 2, \ldots, n, \tag{12}$$

where

$$\begin{aligned}
g'(\eta_j) &:= \max\big\{g'(y_j) \mid y_j \in [\tilde{y}_{j1}^-, \tilde{y}_{j1}^+] \cup [\tilde{y}_{j3}^-, \tilde{y}_{j3}^+]\big\},\\
g'(\xi_j) &:= \max\big\{g'(x_j) \mid x_j \in [\tilde{x}_{j1}^-, \tilde{x}_{j1}^+] \cup [\tilde{x}_{j3}^-, \tilde{x}_{j3}^+]\big\}.
\end{aligned}$$

For $b_{ii}, d_{ii} < 0$, we can define the similar region D_{i1}^-, D_{i1}^+, D_{i3}^-, D_{i3}^+ (see Fig. 3(c), (d)) and there exist a positive constant ϵ_0 and four points $(\tilde{x}_{i1}^-, \tilde{y}_{i1}^+)$, $(\tilde{x}_{i1}^+, \tilde{y}_{i1}^-)$, $(\tilde{x}_{i3}^-, \tilde{y}_{i3}^+)$, $(\tilde{x}_{i3}^+, \tilde{y}_{i3}^-)$, satisfying Eq. (12) and

$$\begin{aligned}
&(\tilde{x}_{i1}^-, \tilde{y}_{i1}^+) \in \mathcal{B}\big((x_{i1}^-, y_{i1}^+), \epsilon_0\big) \cap D_{i1}^-, \qquad && H_i^+(\tilde{x}_{i1}^-) < \tilde{y}_{i1}^+,\ G_i^-(\tilde{y}_{i1}^+) > \tilde{x}_{i1}^-,\\
&(\tilde{x}_{i1}^+, \tilde{y}_{i1}^-) \in \mathcal{B}\big((x_{i1}^+, y_{i1}^-), \epsilon_0\big) \cap D_{i1}^+, && H_i^-(\tilde{x}_{i1}^+) > \tilde{y}_{i1}^-,\ G_i^+(\tilde{y}_{i1}^-) < \tilde{x}_{i1}^+,\\
&(\tilde{x}_{i3}^-, \tilde{y}_{i3}^+) \in \mathcal{B}\big((x_{i3}^-, y_{i3}^+), \epsilon_0\big) \cap D_{i3}^-, && H_i^+(\tilde{x}_{i3}^-) < \tilde{y}_{i3}^+,\ G_i^-(\tilde{y}_{i3}^+) > \tilde{x}_{i3}^-,\\
&(\tilde{x}_{i3}^+, \tilde{y}_{i3}^-) \in \mathcal{B}\big((x_{i3}^+, y_{i3}^-), \epsilon_0\big) \cap D_{i3}^+, && H_i^-(\tilde{x}_{i3}^+) > \tilde{y}_{i3}^-,\ G_i^+(\tilde{y}_{i3}^-) < \tilde{x}_{i3}^+.
\end{aligned}$$

For system (4), consider the following 2^n subset of $\mathcal{C}([-\tau, 0], \mathbb{R}^{2n})$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i = 1$ or 3. Set

$$\Lambda^\alpha = \big\{(\phi, \psi) = (\phi_1, \phi_2, \ldots, \phi_n, \psi_1, \psi_2, \ldots, \psi_n) \,\big|\, (\phi_i, \psi_i) \in \Lambda_i^{\alpha_i}\big\},$$

where

$$\begin{aligned}
\Lambda_i^1 &= \big\{(\phi_i, \psi_i) \in \mathcal{C}\big([-\tau, 0], \mathbb{R}^2\big) \,\big|\, \phi_i(\theta) \in \big[\tilde{x}_{i1}^-, \tilde{x}_{i1}^+\big],\ \psi_i(\theta) \in \big[\tilde{y}_{i1}^-, \tilde{y}_{i1}^+\big],\ \text{for all } \theta \in [-\tau, 0]\big\},\\
\Lambda_i^3 &= \big\{(\phi_i, \psi_i) \in \mathcal{C}\big([-\tau, 0], \mathbb{R}^2\big) \,\big|\, \phi_i(\theta) \in \big[\tilde{x}_{i3}^-, \tilde{x}_{i3}^+\big],\ \psi_i(\theta) \in \big[\tilde{y}_{i3}^-, \tilde{y}_{i3}^+\big],\ \text{for all } \theta \in [-\tau, 0]\big\}.
\end{aligned}$$

Theorem 3.1. *Under assumptions* (H_1) *and* (H_2), *each* Λ^α *is positive invariant with respect to the solution flow generated by system* (4).

Proof. Consider any one of the 2^n subsets of Λ^α. For any initial condition $(\phi, \psi) \in \Lambda^\alpha$, we claim that the solution $(\mathbf{x}(t, \phi, \psi), \mathbf{y}(t, \phi, \psi))$ remains in Λ^α for all $t > 0$. If it is not true, then there exists a component $(x_i(t), y_i(t))$, which is firstly (or one of the first) escaping from Λ_i^1 or Λ_i^3.

Suppose $b_{ii}, d_{ii} > 0$. If $(x_i(t), y_i(t))$ firstly escapes from Λ_i^1, then there exists a $t_0 > 0$, such that $(x_i(t_0), y_i(t_0))$ is on the edge of Λ_i^1 and for any $t < t_0$, $(x_i(t), y_i(t)) \in \Lambda_i^1$. There are four edges of Λ_i^1. If $x_i(t_0) = \tilde{x}_{i1}^-$, $y_i(t_0) \in [\tilde{y}_{i1}^-, \tilde{y}_{i1}^+]$, then

$$\begin{aligned}
\dot{x}_i(t_0) &= -a_i x_i(t_0) + b_{ii} g\big(y_i(t_0 - \tau_{ii})\big) + \sum_{j=1, j\neq i}^n b_{ij} g\big(y_j(t_0 - \tau_{ij})\big) + I_i\\
&\geqslant a_i\left[-\tilde{x}_{i1}^- + \frac{b_{ii}}{a_i} g\big(\tilde{y}_{i1}^-\big) - \frac{1}{a_i}\sum_{j=1, j\neq i}^n |b_{ij}| + \frac{1}{a_i} I_i\right]\\
&= a_i\big[-\tilde{x}_{i1}^- + G_{i1}^-\big(\tilde{y}_{i1}^-\big)\big]\\
&> 0.
\end{aligned}$$

Therefore, $(x_i(t), y_i(t))$ cannot escape from Λ_i^1 through the edge between the points $(\tilde{x}_{i1}^-, \tilde{y}_{i1}^-)$ and $(\tilde{x}_{i1}^-, \tilde{y}_{i1}^+)$. With the similar proof, we can get that $(x_i(t), y_i(t))$ cannot escape from Λ_i^1 through the other three edges. Hence, $(x_i(t), y_i(t))$ cannot escape from Λ_i^1. It can be also proved that $(x_i(t), y_i(t))$ cannot escape from Λ_i^3. So if $b_{ii}, d_{ii} > 0$, each Λ^α are positively invariant of system (4).

For $b_{ii}, d_{ii} < 0$, the proof is similar, and here omit it. From the analysis above, we have the conclusion that under the condition (H_1) and (H_2), each Λ^α are positively invariant of system (3) and (4). The proof is completed. □

Theorem 3.2. *If assumptions* (H_1), (H_2) *and* (H_3) *hold, then there exist* 2^n *exponentially stable equilibria for system* (4).

Proof. Assume $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is the equilibrium in Ω^α for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, with $\alpha_i = 1$ or 3. Consider the single-variable functions $F_i(\cdot)$, $F_i^*(\cdot)$, defined by

$$F_i(s) = a_i - s - \sum_{j=1}^n |b_{ij}| g'(\eta_j) e^{s\tau_{ij}}, \qquad F_i^*(s) = c_i - s - \sum_{j=1}^n |d_{ij}| g'(\xi_j) e^{s\sigma_{ij}},$$

where $g'(\eta_j) := \max\{g'(y_j) |\ y_j \in [\tilde{y}_{j1}^-, \tilde{y}_{j1}^+] \cup [\tilde{y}_{j3}^-, \tilde{y}_{j3}^+]\}$ and $g'(\xi_j) := \max\{g'(x_j) |\ x_j \in [\tilde{x}_{j1}^-, \tilde{x}_{j1}^+] \cup [\tilde{x}_{j3}^-, \tilde{x}_{j3}^+]\}$. Then, $F_i(0) > 0$, $F_i^*(0) > 0$ from inequality (12). Moreover, from the continuity of F_i and F_i^*, there exists a constant $\mu > 0$ such that $F_i(\mu) > 0$ and $F_i^*(\mu) > 0$ for $i = 1, 2, \ldots, n$. Let $(\mathbf{x}(t), \mathbf{y}(t)) = (\mathbf{x}(t; \phi, \psi), \mathbf{y}(t; \phi, \psi))$ be the solution to Eq. (4), with the initial condition $(\phi, \psi) \in \Lambda^\alpha$. Under the transformation $\mathbf{u}(t) = \mathbf{x}(t) - \bar{\mathbf{x}}$, $\mathbf{v}(t) = \mathbf{y}(t) - \bar{\mathbf{y}}$, system (4) becomes

$$\begin{cases}
\dot{u}_i(t) = -a_i u_i(t) + \sum_{j=1}^n b_{ij}[g(v_j(t - \tau_{ij}) + \bar{y}_j) - g(\bar{y}_j)],\\
\dot{v}_i(t) = -c_i v_i(t) + \sum_{j=1}^n d_{ij}[g(x_j(t - \sigma_{ij}) + \bar{x}_j) - g(\bar{x}_j)],
\end{cases} \qquad i = 1, 2, \ldots, n, \tag{13}$$

where $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$. Now, consider function $z_j(\cdot)$ $(j = 1, \dots, 2n)$ defined by

$$z_i(t) = e^{\mu t}\left|u_i(t)\right|, \qquad z_{n+i}(t) = e^{\mu t}\left|v_i(t)\right|, \quad i = 1, 2, \dots, n. \tag{14}$$

Let $\delta > 1$ be an arbitrary real number and denote

$$K := \max_{1\leqslant i\leqslant n}\left\{\sup_{\theta\in[-\tau,0]}\left|x_i(\theta) - \bar{x}_i\right|, \sup_{\theta\in[-\tau,0]}\left|y_i(\theta) - \bar{y}_i\right|\right\} > 0. \tag{15}$$

Hence, $z_j(t) < K\delta$ for $t \in [-\tau, 0]$ and all $j = 1, 2, \dots, 2n$. In the following, we shall prove that

$$z_j(t) < K\delta \quad \text{for all } t > 0,\ j = 1, 2, \dots, 2n. \tag{16}$$

Suppose this is not the case. Then there are an $j \in \{1, 2, \dots, n\}$ (say $j = k$) and a t_0 for the first time such that

$$\begin{aligned}
&z_j(t) \leqslant K\delta, \qquad t \in [-\tau, t_0], \quad j = 1, 2, \dots, 2n,\ j \neq k,\\
&z_k(t) < K\delta, \qquad t \in [-\tau, t_0),\\
&z_k(t_0) = K\delta, \quad \text{with } \dot{z}_k(t_0) \geqslant 0.
\end{aligned}$$

Without losing generality, assume $k \leqslant n$.

Note that $z_k(t_0) = K\delta > 0$ implies $u_k(t_0) \neq 0$. Hence $|u_k(t)|$ and $z_k(t)$ are differentiable at $t = t_0$. From (13), we derive that

$$\frac{d}{dt}\left|u_k(t_0)\right| \leqslant -a_k\left|u_k(t_0)\right| + \sum_{j=1}^{n}|b_{kj}|g'(\eta_j)\left|v_j(t - \tau_{kj})\right|. \tag{17}$$

Hence, from (14) and (17),

$$\begin{aligned}
\dot{z}_k(t_0) &\leqslant \mu e^{\mu t_0}\left|u_k(t_0)\right| + e^{\mu t_0}\left[-a_k\left|u_k(t_0)\right| + \sum_{j=1}^{n}|b_{kj}|g'(\eta_j)\left|v_j(t_0 - \tau_{kj})\right|\right]\\
&\leqslant -(a_k - \mu)e^{\mu t_0}\left|u_k(t_0)\right| + \sum_{j=1}^{n}|b_{kj}|g'(\eta_j)e^{\mu\tau_{kj}}z_{n+j}(t_0 - \tau_{kj})\\
&\leqslant -(a_k - \mu)e^{\mu t_0}\left|u_k(t_0)\right| + \sum_{j=1}^{n}|b_{kj}|g'(\eta_j)e^{\mu\tau_{kj}}\left[\sup_{\theta\in[t_0-\tau,t_0]} z_{n+j}(\theta)\right]\\
&\leqslant -\left\{a_k - \mu - \sum_{j=1}^{n}|b_{kj}|g'(\eta_j)e^{\mu\tau_{kj}}\right\}K\delta\\
&< 0
\end{aligned}$$

which is contradict to $\dot{z}_k(t_0) \geqslant 0$. Hence the inequality (16) holds. Since $\delta > 1$ is arbitrary, by allowing $\delta \to 1^+$, we have $z_i(t) \leqslant K$ for any $t > 0$, $i = 1, 2, \dots, 2n$. Therefore, we have

$$\left|x_i(t) - \bar{x}_i\right| \leqslant Ke^{-\mu t}, \qquad \left|y_i(t) - \bar{y}_i\right| \leqslant Ke^{-\mu t},$$

for any $t > 0$, $i = 1, 2, \dots, n$, $(\mathbf{x}(t), \mathbf{y}(t))$ is exponentially convergent to $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. The proof is completed. □

Remark 3. System (3) can be regarded as a particular case of system (4) for $\tau = 0$. Hence there exist 2^n exponentially stable equilibria for system (3) under the assumptions (H_1), (H_2) and (H_3).

4. Further extension

In this section, we shall consider a more general BAM neural networks as follows:

$$\begin{cases}
\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} g(y_j(t - \tau_{ij})) + I_i,\\
\dot{y}_j(t) = -c_j y_j(t) + \sum_{i=1}^{n} d_{ji} g(x_i(t - \sigma_{ji})) + J_j,
\end{cases} \tag{18}$$

for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, where $g(x) = \tanh(x)$. If all $\tau_{ij}, \sigma_{ji} = 0$, then system (18) is an ordinary differential equations; else it is a functional differential equations. If $n = m$, system (18) is equal to system (4). If $n \neq m$, then the number of neurons in each layer of the BAM neural networks are different. To consider the multistability of system (18), we propose more general conditions as follows, for $0 \leqslant k \leqslant \min\{n, m\}$

$$(\mathrm{H}_1^{(k)}) \quad \frac{b_{ii}d_{ii}}{a_ic_i} > 1, \quad i = 1, 2, \ldots, k$$

$(\mathrm{H}_2^{(k)})$ for $i = 1, 2, \ldots, k$, there exist two points (u_{i1}, v_{i1}) and (u_{i2}, v_{i2}), where $u_{i1} < u_{i2}$, that

$$\begin{cases} \text{if } b_{ii}, d_{ii} > 0, & H_i^+(u_{i1}) < v_{i1}, G_i^+(v_{i1}) < u_{i1}, \\ & H_i^-(u_{i2}) > v_{i2}, G_i^-(v_{i2}) > u_{i2}; \\ \text{if } b_{ii}, d_{ii} < 0, & H_i^-(u_{i1}) > v_{i1}, G_i^+(v_{i1}) < u_{i1}, \\ & H_i^+(u_{i2}) < v_{i2}, G_i^-(v_{i2}) > u_{i2}, \end{cases}$$

where

$$G_i^+(y_i) = \frac{1}{a_i}\left(b_{ii}g\big(y_i(t)\big) + I_i + \sum_{j=1, j\neq i}^{m} |b_{ij}|\right), \qquad G_i^-(y_i) = \frac{1}{a_i}\left(b_{ii}g\big(y_i(t)\big) + I_i - \sum_{j=1, j\neq i}^{m} |b_{ij}|\right),$$
$$H_j^+(x_j) = \frac{1}{c_i}\left(d_{ii}g\big(x_i(t)\big) + J_i + \sum_{j=1, j\neq i}^{n} |d_{ij}|\right), \qquad H_i^-(x_i) = \frac{1}{c_i}\left(d_{ii}g\big(x_i(t)\big) + J_i - \sum_{j=1, j\neq i}^{n} |d_{ij}|\right).$$

As in Section 2, there also exists the similar points (x_{i1}^+, y_{i1}^+), (x_{i1}^-, y_{i1}^-), (x_{i2}^+, y_{i2}^+), (x_{i2}^-, y_{i2}^-), (x_{i3}^+, y_{i3}^+), (x_{i3}^-, y_{i3}^-) and the similar regions

$$\Omega_i^1 := \big\{(x, y) \in \mathbb{R}^2 \bigm| (x, y) \in \big[x_{i1}^-, x_{i1}^+\big] \times \big[y_{i1}^-, y_{i1}^+\big]\big\},$$
$$\Omega_i^2 := \big\{(x, y) \in \mathbb{R}^2 \bigm| (x, y) \in \big[x_{i2}^-, x_{i2}^+\big] \times \big[y_{i2}^-, y_{i2}^+\big]\big\},$$
$$\Omega_i^3 := \big\{(x, y) \in \mathbb{R}^2 \bigm| (x, y) \in \big[x_{i3}^-, x_{i3}^+\big] \times \big[y_{i3}^-, y_{i3}^+\big]\big\},$$

where $i \leqslant k$. For $i > k$, denote

$$x_{i0}^- = \frac{1}{a_i}\left(-\sum_{j=1}^{m} |b_{ij}| + I_i\right), \qquad x_{i0}^+ = \frac{1}{a_i}\left(\sum_{j=1}^{m} |b_{ij}| + I_i\right), \quad i = k+1, \ldots, n,$$
$$y_{i0}^- = \frac{1}{c_i}\left(-\sum_{j=1}^{n} |d_{ij}| + J_i\right), \qquad y_{i0}^+ = \frac{1}{c_i}\left(\sum_{j=1}^{n} |d_{ij}| + J_i\right), \quad i = k+1, \ldots, m$$

and

$$\Omega^0 := \big\{(x_{k+1}, \ldots, x_n, y_{k+1}, \ldots, y_m) \in \mathbb{R}^{(n-k)\times(m-k)} \bigm| x_i \in \big[x_{i0}^-, x_{i0}^+\big], y_j \in \big[y_{j0}^-, y_{j0}^+\big] \text{ for } i = k+1, \ldots, n,\ j = k+1, \ldots, m\big\}.$$

There are 3^k disjoint closed regions in $\mathbb{R}^{n\times m}$. Set $\Omega^\beta = \{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{n\times m} | (x_i, y_i) \in \Omega_i^{\beta_i}, (x_{k+1}, \ldots, x_n, y_{k+1}, \ldots, y_m) \in \Omega^0\}$ with $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$, and $\beta_i = 1, 2, 3$.

Applied the same method in Section 3, the similar points $(\tilde{x}_{i1}^-, \tilde{y}_{i1}^-)$, $(\tilde{x}_{i1}^+, \tilde{y}_{i1}^+)$, $(\tilde{x}_{i3}^-, \tilde{y}_{i3}^-)$, $(\tilde{x}_{i3}^+, \tilde{y}_{i3}^+)$ can be found for $i \leqslant k$. Consider the following 2^k subsets of $\mathcal{C}([-\tau, 0], R^{n\times m})$. Let $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ with $\beta_i = 1$, or 3, and set

$$\Lambda^\beta = \big\{(\phi, \psi) = (\phi_1, \phi_2, \ldots, \phi_n, \psi_1, \psi_2, \ldots, \psi_n) \bigm| (\phi_i, \psi_i) \in \Lambda_i^{\beta_i}, \quad \text{for } i \leqslant k;\ (\phi_{k+1}, \ldots, \phi_n, \psi_{k+1}, \ldots, \psi_m) \in \Lambda^0\big\},$$

where

$$\Lambda_i^1 = \big\{(\phi_i, \psi_i) \in \mathcal{C}\big([-\tau, 0], \mathbb{R}^2\big) \bigm| \phi_i(\theta) \in \big[\tilde{x}_{i1}^-, \tilde{x}_{i1}^+\big],\ \psi_i(\theta) \in \big[\tilde{y}_{i1}^-, \tilde{y}_{i1}^+\big],\ \text{for all } \theta \in [-\tau, 0]\big\},$$
$$\Lambda_i^3 = \big\{(\phi_i, \psi_i) \in \mathcal{C}\big([-\tau, 0], \mathbb{R}^2\big) \bigm| \phi_i(\theta) \in \big[\tilde{x}_{i3}^-, \tilde{x}_{i3}^+\big],\ \psi_i(\theta) \in \big[\tilde{y}_{i3}^-, \tilde{y}_{i3}^+\big],\ \text{for all } \theta \in [-\tau, 0]\big\},$$
$$\Lambda^0 = \big\{(\phi_{k+1}, \ldots, \phi_n, \psi_{k+1}, \ldots, \psi_m) \in \mathcal{C}\big([-\tau, 0], \mathbb{R}^{(n-k)\times(m-k)}\big) \big| \phi_i(\theta) \in \big[x_{i0}^-, x_{i0}^+\big],$$
$$\psi_i(\theta) \in \big[y_{j0}^-, y_{j0}^+\big],\ \text{for all } \theta \in [-\tau, 0]\big\}.$$

Consider $(\mathrm{H}_3^{(k)})$:

$$\begin{cases} -a_i + \sum_{j=1}^k |b_{ij}|g'(\eta_j) + \sum_{j=k+1}^m |b_{ij}| < 0, & g'(\eta_j) := \max\{g'(y_j) \mid y_j \in [y_{j1}^-, y_{j1}^+] \cup [y_{j3}^-, y_{j3}^+]\}, \\ -c_i + \sum_{j=1}^k |d_{ij}|g'(\xi_j) + \sum_{j=k+1}^n |d_{ij}| < 0, & g'(\xi_j) := \max\{g'(x_j) \mid x_j \in [x_{j1}^-, x_{j1}^+] \cup [x_{j3}^-, x_{j3}^+]\}. \end{cases}$$

Theorem 4.1. *Under conditions* $(\mathrm{H}_1^{(k)})$ *and* $(\mathrm{H}_2^{(k)})$*, there exist* 3^k *equilibria in system* (18).

Theorem 4.2. *Under conditions* $(\mathrm{H}_1^{(k)})$ *and* $(\mathrm{H}_2^{(k)})$*, there exist* 2^k *positive invariant sets, denoted by* Λ^β*, in system* (18).

Theorem 4.3. *If conditions* $(\mathrm{H}_1^{(k)})$*,* $(\mathrm{H}_2^{(k)})$*, and* $(\mathrm{H}_3^{(k)})$ *hold, there exist* 2^k *exponentially stable equilibria for system* (18).

The proofs are similar with Theorem 2.2, Theorem 3.1 and Theorem 3.2, and here omit them.

Remark 4. If $n = m$ and $k = n$, the conclusion in Theorem 4.3 is the same with Theorem 3.2.

Remark 5. If $k = 0$, the condition $(\mathrm{H}_1^{(0)})$, $(\mathrm{H}_2^{(0)})$, and $(\mathrm{H}_3^{(0)})$ can be rewritten as follows

$$\begin{cases} -a_i + \sum_{j=1}^{m} |b_{ij}| < 0, & i = 1, \dots, n, \\ -c_i + \sum_{j=1}^{n} |d_{ij}| < 0, & i = 1, \dots, m. \end{cases}$$

Use the knowledge of M-matrix in [24], there exists an equilibrium for system (18), which is global exponentially stable.

5. Numerical examples

In this section, two numerical examples are given to illustrate the validity of results.

Example 1. Consider the BAM neural network as follows,

$$\begin{cases} \dot{x_1}(t) = -x_1(t) - 2g(y_1(t-\tau)) + 0.2g(y_2(t-\tau)) + 0.5, \\ \dot{x_2}(t) = -x_2(t) + g(y_1(t-\tau)) + 4g(y_2(t-\tau)) - 1, \\ \dot{y_1}(t) = -y_1(t) - 3g(x_1(t-\tau)) - 0.5g(x_2(t-\tau)) - 0.2, \\ \dot{y_2}(t) = -y_2(t) + g(x_1(t-\tau)) + 2.5g(x_2(t-\tau)) + 0.3, \end{cases} \tag{19}$$

where $g(s) = \tanh(s)$, and $\dot{g}(s) = 1 - g^2(s)$. Hence,

$$\begin{aligned} &G_1^+(y_1) = -2g\big(y_1(t)\big) + 0.7, && G_1^-(y_1) = -2g\big(y_1(t)\big) + 0.3, \\ &G_2^+(y_2) = 4g\big(y_2(t)\big), && G_2^-(y_2) = 4g\big(y_2(t)\big) - 2, \\ &H_1^+(x_1) = -3g\big(x_1(t)\big) + 0.3, && H_1^-(x_1) = -3g\big(x_1(t)\big) - 0.7, \\ &H_2^+(x_2) = 2.5g\big(x_1(t)\big) + 1.3, && H_2^-(x_2) = 2.5g\big(x_2(t)\big) - 0.7. \end{aligned}$$

Herein, the parameters satisfy our conditions:

Condition (H_1):

$$\frac{b_{11}d_{11}}{a_1c_1} = 6 > 1, \qquad \frac{b_{22}d_{22}}{a_2c_2} = 10 > 1.$$

Condition (H_2): there exist four points $(u_{11}, v_{11}) = (-1, 1.4)$, $(u_{12}, v_{12}) = (1, -1)$, $(u_{21}, v_{21}) = (-2, -1)$, $(u_{22}, v_{22}) = (1, 1)$, where $u_{11} < u_{12}$ and $u_{21} < u_{22}$, that

$$\begin{aligned} &H_1^-(u_{11}) = 1.5848 > v_{11}, && H_2^+(u_{21}) = -1.1101 < v_{21}, \\ &G_1^+(v_{11}) = -1.0707 < u_{11}, && G_2^+(v_{21}) = -3.0464 < u_{21}, \\ &H_1^+(u_{12}) = -1.9848 < v_{12}, && H_2^-(u_{22}) = 1.2040 > v_{22}, \\ &G_1^-(v_{12}) = 1.8232 > u_{12}, && G_2^-(v_{22}) = 1.0464 > u_{22}. \end{aligned}$$

Condition (H_3):

$$\begin{cases} -1 + 2g'(\eta_1) + 0.2g'(\eta_2) = -0.3577 < 0, \\ -1 + g'(\eta_1) + 4g'(\eta_2) = -0.2011 < 0, \\ -1 + 3g'(\xi_1) + 0.5g'(\xi_2) = -0.5314 < 0, \\ -1 + g'(\xi_1) + 2.5g'(\xi_2) = -0.1244 < 0, \end{cases}$$

where

$$x_{11}^- = -1.6919, \qquad x_{11}^+ = -1.1923, \qquad x_{13}^- = 2.2796, \qquad x_{13}^+ = 2.6974, \qquad g'(\eta_1) = g'\big(x_{11}^+\big) = 0.3089,$$

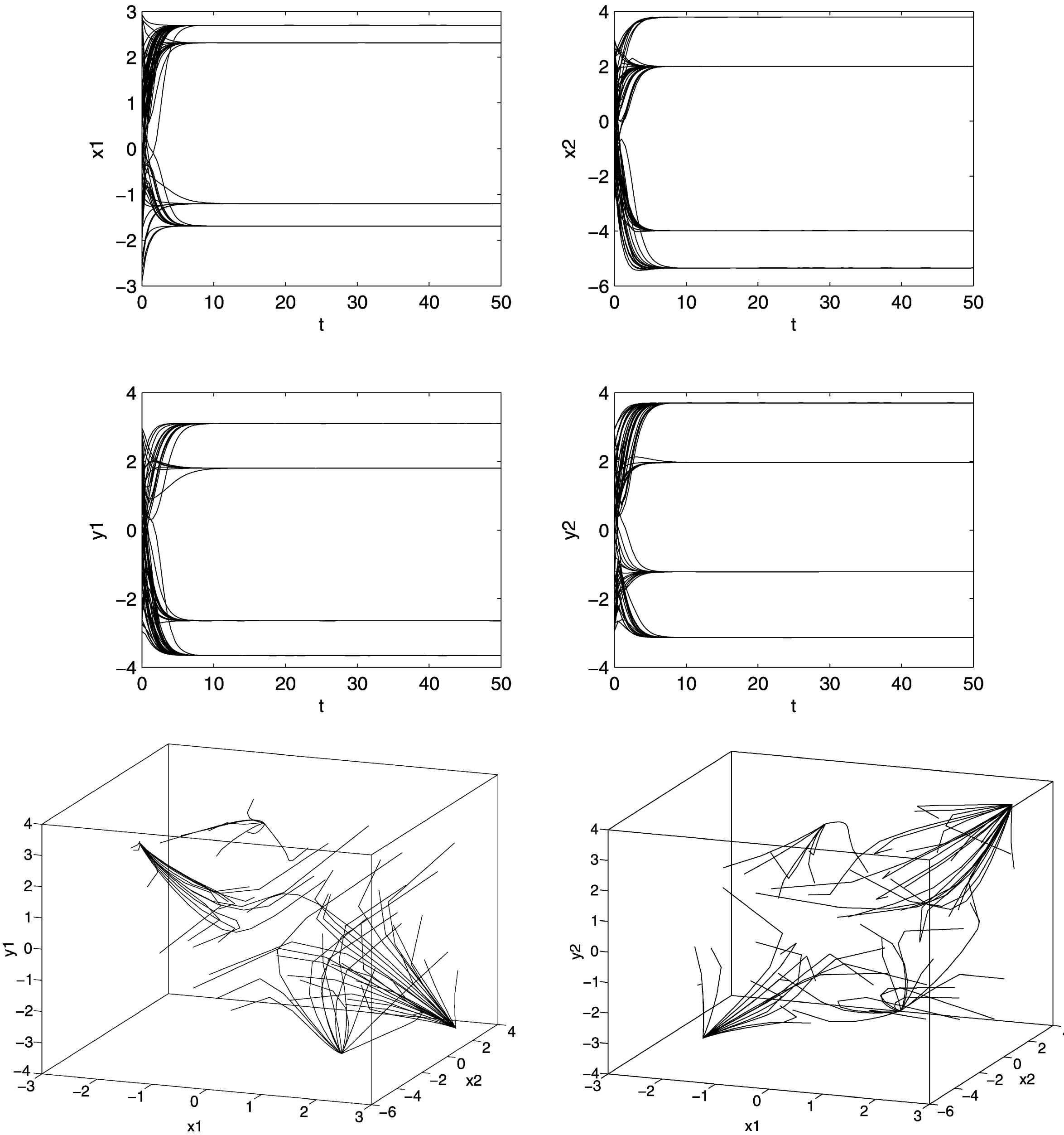


Fig. 4. Transient behavior of system (19), where $\tau = 0$.

$$y_{11}^- = 1.7939, \quad y_{11}^+ = 3.1031, \quad y_{13}^- = -3.6728, \quad y_{13}^+ = -2.6378, \quad g'(\xi_1) = g'\big(y_{11}^-\big) = 0.1048,$$
$$x_{21}^- = -5.9867, \quad x_{21}^+ = -3.3267, \quad x_{23}^- = 1.7109, \quad x_{23}^+ = 3.9959, \quad g'(\eta_2) = g'\big(x_{23}^-\big) = 0.1225,$$
$$y_{21}^- = -3.2000, \quad y_{21}^+ = -1.1936, \quad y_{23}^- = 1.6419, \quad y_{23}^+ = 3.7983, \quad g'(\xi_2) = g'\big(y_{21}^+\big) = 0.3083.$$

For $\tau = 0$, system (19) has 9 equilibria, in which 4 equilibria are stable, according to Theorem 3.2. Simulation results with 60 random initial states are depicted in Fig. 4.

Example 2. In Example 1, if we delete the neuron y_2 and its corresponding connections with other neurons, then the neural networks is as follows:

$$\begin{cases} \dot{x}_1(t) = -x_1(t) - 2g(y_1(t-\tau)) + 0.5, \\ \dot{x}_2(t) = -x_2(t) + g(y_1(t-\tau)) - 1, \\ \dot{y}_1(t) = -y_1(t) - 3g(x_1(t-\tau)) - 0.5g(x_2(t-\tau)) - 0.2, \end{cases} \tag{20}$$

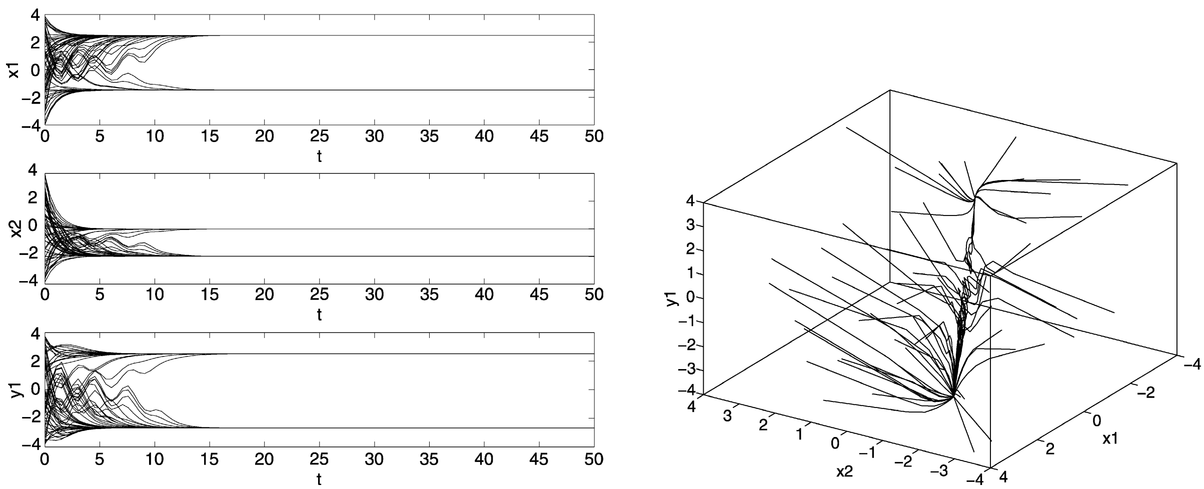


Fig. 5. Transient behavior of system (20), where $\tau = 1$.

where $g(s) = \tanh(s)$, and $\dot{g}(s) = 1 - g^2(s)$. Hence,

$$G_1^+(y_1) = -2g\big(y_1(t)\big) + 0.5, \qquad G_1^-(y_1) = -2g\big(y_1(t)\big) + 0.5,$$
$$H_1^+(x_1) = -3g\big(x_1(t)\big) + 0.3, \qquad H_1^-(x_1) = -3g\big(x_1(t)\big) - 0.7,$$

where $G_1^+(y_1) = G_1^-(y_1)$.

Herein, the parameters satisfy our conditions:

Condition $(\mathrm{H}_1^{(1)})$:

$$\frac{b_{11}d_{11}}{a_1c_1} = 6 > 1.$$

Condition $(\mathrm{H}_2^{(1)})$: there exist four points $(u_{11}, v_{11}) = (-1, 1.4)$, $(u_{12}, v_{12}) = (1, -1)$, where $u_{11} < u_{12}$ that

$$H_1^-(u_{11}) = 1.5848 > v_{11}, \qquad H_1^+(u_{12}) = -1.9848 < v_{12},$$
$$G_1^+(v_{11}) = -1.2707 < u_{11}, \qquad G_1^-(v_{12}) = 2.0232 > u_{12}.$$

Condition $(\mathrm{H}_3^{(1)})$:

$$\begin{cases} -1 + 2g'(\eta_1) = -0.5858 < 0, \\ -1 + g'(\eta_1) = -0.7929 < 0, \\ -1 + 3g'(\xi_1) + 0.5 = -0.2759 < 0, \end{cases}$$

where

$$x_{11}^- = -1.490, \quad x_{11}^+ = -1.424, \quad x_{13}^- = 2.480, \quad x_{13}^+ = 2.497, \quad g'(\eta_1) = g'\big(x_{11}^+\big) = 0.2071,$$
$$y_{11}^- = 1.971, \quad y_{11}^+ = 3.010, \quad y_{13}^- = -3.660, \quad y_{13}^+ = -2.658, \quad g'(\xi_1) = g'\big(y_{11}^-\big) = 0.0747.$$

For $\tau = 1$, from Theorem 4.3, there are two equilibria which are exponentially stable in system (20). It is the same with the condition for $\tau = 0$ in system (20). And the dynamics are shown in Fig. 5.

6. Conclusions

In this Letter, the multistability has been studied for BAM neural networks. The capacity of the associative memories in the BAM neural networks are learned. Due to the loss of self-connection, the amount of the equilibria is not as large as other neural networks in [14–16]. For the BAM network with n neurons on each layer, there exist 2^n exponentially stable equilibria. Moreover, the model has been extended to more general condition. If there are n and m neurons on the two layers respectively, some sufficient conditions are proposed to warrant the existence of 2^k exponentially stable equilibria, where k varies from 0 to $\min\{n, m\}$. For $k = 0$, the equilibrium is global exponentially stable under the proposed condition. Finally, some examples have been provided to verify the new results. Furthermore, the coexistence of stable equilibria, stable limit cycles, and even chaos is an interesting topic. It will be investigated in near future.

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