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Multistability of neural networks with discontinuous activation function ☆

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Abstract

In this paper, the multistability is studied for two-dimensional neural networks with multilevel activation functions. And it is showed that the system has n^2 isolated equilibrium points which are locally exponentially stable, where the activation function has n segments. Furthermore, evoked by periodic external input, n^2 periodic orbits which are locally exponentially attractive, can be found. And these results are extended to k-neuron networks, which is really enlarge the capacity of the associative memories. Examples and simulation results are used to illustrate the theory.

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Keywords: Neural networks; Multilevel function; Activation function; Multistability

1. Introduction

In the past decades, the studies of neural networks have attracted a tremendous amount of research interest. The dynamical behaviors including stability [1–5], periodic bifurcation and chaos [6–9] of the neural networks have become a focal topic. While the applications of the neural networks range from classifications, associative memory, image processing, and pattern recognition to parallel computation and its ability to solve optimization problems. While the theory on the dynamics of the networks have been developed according to the purposes of the applications.

In some applications, there is a need to design a neural circuit possessing a unique equilibrium point. For example, when solving important classes of optimization problems [10–12], where uniqueness of the equilibrium is required to prevent convergence toward local minima (undesired spurious responses) and hence ensure global optimization. Such a convergent behavior is referred to as "monostability" of a network. Many results on global convergence concern neural networks where the neuron activations are modeled by Lipschitz-continuous

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functions. However, discontinuous neuron activations are of importance and do frequently arise in practice. For example, the classical Hopfield neural networks (HNNs) with graded response neurons [13]. The dynamical behaviors of neural networks with discontinuous activation functions have been studied in [14–16].

On the other hand, when a neural network is employed as an associative memory storage for pattern recognition, the existence of many equilibria is a necessary feature. The notion of ‘‘multistability’’ of a neural network is used to describe coexistence of multiple stable patterns such as equilibria or periodic orbits. The existence of multiple stable patterns has been developed for cellular neural networks in [17–19]. It is found that an k-neuron cellular neural networks can have up to 2^k locally stable equilibria in [20]; and 2^k locally attractive periodic orbits with periodic external inputs in [21]. Some similar results have been found with Hopfield-type neuron activations in [22]. The multistability of neural networks with piecewise linear activation functions has developed in [23,24]. In this paper, we study a type of two-dimensional neural networks with discontinuous neuron activations, which can have n^2 locally stable equilibria, where n is the number of segments of the multilevel activation functions. And n^2 locally attractive periodic orbits can been found with periodic external inputs. In extension, there could been n^k locally stable equilibria in a k-neuron networks. Compared with the previous result [20–24], by using multilevel activation function, we can design neural networks with arbitrary number of stable equilibria which is really enlarge the capacity of associative memories.

The remaining part of this paper is organized as follows. In Section 2 the model and the activation function are given. In Section 3, the number of equilibria of neural networks are obtained. In Section 4, three illustrative examples are provided with simulation results. Finally, conclusions are given in Section 5.

2. Model description

Consider two-dimensional (2-D) neural networks described by the following of differential equations:

$$\begin{cases} \frac{dx_1(t)}{dt} = -x_1(t) + a_{11}f(x_1(t)) + a_{12}f(x_2(t)) + I_1, \\ \frac{dx_2(t)}{dt} = -x_2(t) + a_{21}f(x_1(t)) + a_{22}f(x_2(t)) + I_2 \end{cases} \tag{1}$$

or its equivalent vector form

$$\frac{dx(t)}{dt} = -x(t) + Af(x(t)) + I,$$

where x_i denotes the activity neuron i, $x = (x_1, x_2)^T \in \mathbb{R}^2$ denotes neuron state, $f(x) = (f(x_1), f(x_2))^T$ denotes activation function, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$ is a matrix whose entries represent the synaptic neuron interconnections, and $I = (I_1(t), I_2(t))^T \in \mathbb{R}^2$ is a vector of constant external neuron inputs. If the inputs are ω-periodic, then the neural networks can be written as follows:

$$\begin{cases} \frac{dx_1(t)}{dt} = -x_1(t) + a_{11}f(x_1(t)) + a_{12}f(x_2(t)) + I_1(t), \\ \frac{dx_2(t)}{dt} = -x_2(t) + a_{21}f(x_1(t)) + a_{22}f(x_2(t)) + I_2(t), \end{cases} \tag{2}$$

where the inputs $I(t) = (I_1(t), I_2(t))^T \in \mathbb{R}^2$ is a vector with ω-period.

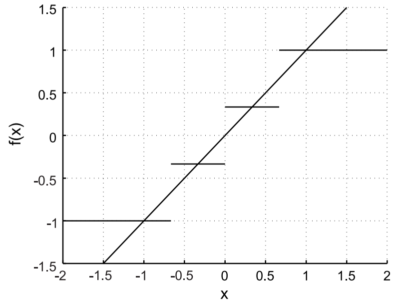
In the neural networks, Eqs. (1) and (2), the activation function is discontinuous, which has n segments. Choose two arrays of number $\{b_0, b_1, b_2, \ldots, b_n\}$, $\{c_1, c_2, \ldots, c_n\}$ as

$$-1 = c_1 < b_1 < c_2 < b_2 < c_3 < \ldots < b_{n-2} < c_{n-1} < b_{n-1} < c_n = 1; \quad \text{and} \quad b_0 = -\infty, b_n = +\infty.$$

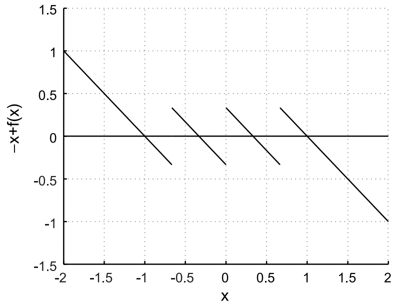
$$f(x) = \begin{cases} c_1, & x < b_1; \\ c_i, & b_{i-1} \leqslant x < b_i; \\ c_n, & x \geqslant b_{n-1}. \end{cases} \tag{3}$$

for $i = 1, 2, \ldots, n$; while $x < b_1 \iff b_0 \leqslant x < b_1$, and $x \geqslant b_{n-1} \iff b_{n-1} \leqslant x < b_n$. So f can be rewritten as:

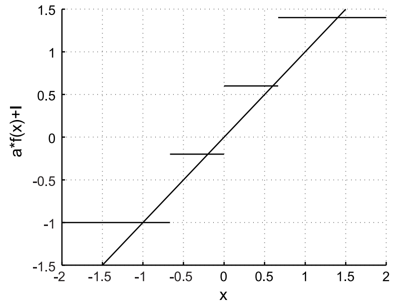
$$f(x) = c_i, \quad \text{if } b_{i-1} \leqslant x < b_i.$$



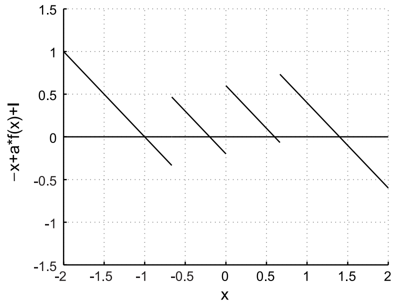
(a) The figure of $f(x)$, which is a multilevel function with four segments, and cross line $g(x) = x$ four times.



(b) The figure of $-x + f(x)$, where $f(x)$ is a multilevel function with four segments. And $-x + f(x)$ have four zero points.



(c)The figure of $af(x)$, where $f(x)$ is a multilevel function with four segments, and cross line $g(x) = x$ four times.



(d)The figure of $-x + af(x)$, where $f(x)$ is a multilevel function with four segments. And $-x + af(x)$ have four zero points.

Fig. 1. The figure of $f(x)$, $-x + f(x)$, $af(x)$ and $-x + af(x)$, where $f(x)$ is a multilevel function with four segments.

3. Main results

First, we consider the following model with the single neuron:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + f(x(t)) = F(x(t)). \tag{4}$$

Denote $N_i = (b_{i-1}, b_i)$. $f(x)$, as defined above, has n points of intersection with line $g(x) = x$, which are $c_1, c_2, \ldots, c_n$, $c_i \in N_i$ (See Fig. 1(a)). Hence there are n zero points for $F(x)$ (see Fig. 1b). Furthermore, if $x(t_0) \in [c_i - \varepsilon, c_i + \varepsilon] \subset N_i$, then the Eq. (4) can be rewritten as follows:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + c_i.$$

Hence, c_i is an equilibria of the equation, which is locally stable. So the Eq. (4) has n locally stable equilibrium, which are $c_1, c_2, \ldots, c_n$. Similarly, consider the equation as follows:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + af(x(t)) + I. \tag{5}$$

If $af(x) + I$ has n points of intersection with line $g(x) = x$, which is equivalent to $(ac_i + I) \in N_i$ (see Fig. 1c), then Eq. (5) has n locally stable equilibrium (see Fig. 1d).

For example in Fig. 1, $a = 1.2$, $I = 0.2$ and $f(x)$ is defined as follows:

$$f(x) = \begin{cases} -1, & x < -\frac{2}{3}; \\ -\frac{1}{3}, & -\frac{2}{3} \leqslant x < 0; \\ \frac{1}{3}, & 0 \leqslant x < \frac{2}{3}; \\ 1, & x \geqslant \frac{2}{3}. \end{cases} \tag{6}$$

From Fig. 1a and c in the left, we can find that $f(x)$ and $af(x) + I$ both have four points of intersection with line $g(x) = x$ four times. So Fig. 1b and d show that $-x + f(x)$ and $-x + af(x) + I$ both have four zero points. Hence, Eqs. (4), (5) have four locally stable equilibria.

In the following, two-dimensional neural networks are considered. We will give two theorems for checking the multistability of system (1) and (2).

Theorem 1. *The two-dimensional neural network* (1) *has* n^2 *locally exponentially stable equilibrium points with constant input I and n segments multilevel function f, if for* $\forall\ i \in 1, 2, \ldots, n$,

$$\begin{cases} b_{i-1} + |a_{12}| < a_{11}c_i + I_1 < b_i - |a_{12}| \\ b_{i-1} + |a_{21}| < a_{22}c_i + I_2 < b_i - |a_{21}| \end{cases} \tag{7}$$

Proof 1. Denote $N_{ij} = (b_{i-1}, b_i) \times (b_{j-1}, b_j)$. If $(x^*_{1i}, x^*_{2j}) \in N_{ij}$ is an equilibrium of system (1), then we have

$$\begin{cases} -x^*_{1i} + a_{11}f(x^*_{1i}) + a_{12}f(x^*_{2j}) + I_1 = 0, \\ -x^*_{2j} + a_{21}f(x^*_{1i}) + a_{22}f(x^*_{2j}) + I_2 = 0, \end{cases}$$

$$\begin{cases} x^*_{1i} = a_{11}c_i + a_{12}c_j + I_1, \\ x^*_{2j} = a_{21}c_i + a_{22}c_j + I_2. \end{cases}$$

From Eq. (7), for $\forall i, j \in \{1, 2, \ldots, n\}$, we have

$$b_{i-1} < a_{11}c_i + a_{12}c_j + I_1 < b_i\ ,$$
$$b_{j-1} < a_{21}c_i + a_{22}c_j + I_2 < b_j.$$

Hence, in the local area N_{ij}, system (1) has exactly equilibrium (x^*_{1i}, x^*_{2j}), where $x^*_{1i} = a_{11}c_i + a_{12}c_j + I_1 \in (b_{i-1}, b_i), x^*_{2j} = a_{21}c_i + a_{22}c_j + I_2 \in (b_{j-1}, b_j)$. So, for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$, there are n^2 local equilibrium points in all.

Next, we prove the n^2 equilibrium points are local exponentially stable. For $\forall$ N_{ij}, we can find $b'_{i-1}, b'_i, b'_{j-1}, b'_j$, which have

$$b_{i-1} < b'_{i-1} < x^*_{1i} < b'_i < b_i,\ b_{j-1} < b'_{j-1} < x^*_{2j} < b'_j < b_j.$$

Denote $N'_{ij} = [b'_{i-1}, b'_i] \times [b'_{j-1}, b'_j]$, which is a close set. For $\forall (x_1(t_0), x_2(t_0)) \in N_{ij}$, there must be a set N'_{ij}, which contain $(x_1(t_0), x_2(t_0))$. We say that it stays in the local area N'_{ij}. If this is not true, then there is a time $t > t_0$, $(x_1(t), x_2(t))$ not in N'_{ij}. There must be $x_1(t)$ not in $[b'_{i-1}, b'_i]$, or $x_2(t)$ not in $[b'_{j-1}, b'_j]$. If $x_1(t) < b'_{i-1}$, then there is a time $t_1 \leqslant t$ that $x_1(t_1) = b'_{i-1}$ and $\frac{dx_1(t)}{dt}|_{t=t_1} < 0$. From the Eq. (1), we have

$$\left.\frac{dx_1(t)}{dt}\right|_{t=t_1} = -x_1(t_1) + a_{11}f(x_1(t_1)) + a_{12}f(x_2(t_1)) + I_1 = -b'_{i-1} + a_{11}c_i + a_{12}f(x_2(t_1)) + I_1 > 0,$$

which is contradiction. So we have $x_1(t) \geqslant b'_{i-1}$ for all $t > 0$. In the same way, we can prove $x_1(t) \leqslant b'_i$ for all $t < 0$, and $x_2(t) \in [b'_{j-1}, b'_j]$. This shows the orbits stay in the area N'_{ij}, which is also in N_{ij}. Now we can rewrite Eq. (1) as

$$\begin{cases} \frac{dx_1(t)}{dt} = -x_1(t) + x^*_{1i}, \\ \frac{dx_2(t)}{dt} = -x_2(t) + x^*_{2j}. \end{cases}$$

Hence, the orbits from N_{ij} tend to (x^*_{1i}, x^*_{2j}) exponentially. There are up to n^2 locally exponentially stable equilibrium points $\{(x^*_{1i}, x^*_{2j})\}$, $i, j = 1, 2, \ldots, n$. and the proof is completed. □

Theorem 2. *The two-dimensional neural network (2) has n^2 locally exponentially limit cycles with ω-periodic input $I(t)$ and n segments multilevel function f, if for $\forall i \in 1, 2, \ldots, n$,*

$$\begin{cases} b_{i-1} + |a_{12}| < a_{11}c_i + I_1(t) < b_i - |a_{12}|, \\ b_{i-1} + |a_{21}| < a_{22}c_i + I_2(t) < b_i - |a_{21}|. \end{cases} \tag{8}$$

Proof 2. With the similar proof in Theorem 1, we can derive the same conclusion. For $\forall\ i, j = 1, 2, \ldots, n$, if $(x_1(t_0), x_2(t_0)) \in N_{ij}$, there must be a close set

$$N'_{ij} = [b'_{i-1}, b'_i] \times [b'_{j-1}, b'_j] \subset N_{ij}$$

and the trajectory, from the initial value $(x_1(t_0), x_2(t_0)) \in N'_{ij}$, will remain in the local area N'_{ij}. Hence it will also remain in N_{ij}. Eq. (2) can be rewritten as follows:

$$\begin{cases} \frac{dx_1(t)}{dt} = -x_1(t) + a_{11}c_i + a_{12}c_j + I_1(t), \\ \frac{dx_2(t)}{dt} = -x_2(t) + a_{21}c_i + a_{22}c_j + I_2(t). \end{cases} \tag{9}$$

Let $x(t; t_0, \tilde{x}_0)$, and $x(t; t_0, \hat{x}_0)$ be two states of Eq. (2), with initial conditions $(t_0, \tilde{x}_0)$ and $(t_0, \hat{x}_0)$, where $\tilde{x}_0$, $\hat{x}_0 \in N_{ij}$. We can find a close set $N'_{ij} \subset N_{ij}$, such that $\tilde{x}_0$, $\hat{x}_0 \in N'_{ij}$. From Eq. (5), for $t > t_0, i = 1, 2$

$$\frac{d(x_i(t; t_0, \tilde{x}_0) - x_i(t; t_0, \hat{x}_0))}{dt} = -(x_i(t; t_0, \tilde{x}_0) - x_i(t; t_0, \hat{x}_0)). \tag{10}$$

Define a mapping H: $\mathbb{R}^2 \to \mathbb{R}^2$ by $H(s) = x(t_0 + \omega; t_0, s)$. Then $H(N'_{ij}) \subset N'_{ij}$, and $H^m(s) = x(t_0 + m\omega; t_0, s)$. We can choose a positive m such that $\exp(-m\omega) \leqslant \alpha < 1$, Hence, from Eq. (10),

$$\|H^m(\tilde{x}_0) - H^m(\hat{x}_0)\| \leqslant \exp(-m\omega)\|\tilde{x}_0 - \hat{x}_0\| \leqslant \alpha\|\tilde{x}_0 - \hat{x}_0\|.$$

By contraction mapping principle, there exists a unique fixed point $x^* \in N'_{ij}$ such that $H^m(x^*) = x^*$. Obviously, it is also the unique fixed point in N_{ij}. In addition, $H^m(H(x^*)) = H(H^m(x^*)) = H(x^*)$. This shows that $H(x^*)$ is also a fixed point of H^m. Hence, by the uniqueness of the fixed point of the mapping H^m, $H(x^*) = x^*$; that is $x(t_0 + \omega; t_0, x^*) = x^*$. Let $x(t; t_0, x^*)$ be a state of Eq. (2), with initial condition (t_0, x^*). Then from Eq. (9), for any $t > 0$

$$\frac{dx_1(t; t_0, x^*)}{dt} = -x_1(t; t_0, x^*) + a_{11}c_i + a_{12}c_j + I_1(t),$$
$$\frac{dx_2(t; t_0, x^*)}{dt} = -x_2(t; t_0, x^*) + a_{21}c_i + a_{22}c_j + I_2(t).$$

Hence, for any $t + \omega \geqslant t_0$

$$\begin{aligned} \frac{dx_1(t + \omega; t_0, x^*)}{dt} &= -x_1(t + \omega; t_0, x^*) + a_{11}c_i + a_{12}c_j + I_1(t + \omega) \\ &= -x_1(t + \omega; t_0, x^*) + a_{11}c_i + a_{12}c_j + I_1(t), \\ \frac{dx_2(t + \omega; t_0, x^*)}{dt} &= -x_2(t + \omega; t_0, x^*) + a_{21}c_i + a_{22}c_j + I_2(t). \end{aligned}$$

This implies that $x(t + \omega; t_0, x^*)$ is also a state of Eq. (2), with initial condition (t_0, x^*). $x(t_0 + \omega; t_0, x^*) = x^*$ implies that for any $t > 0$

$$x(t + \omega; t_0, x^*) = x(t; t_0 + \omega, x^*) = x(t; t_0, x^*).$$

$x(t; t_0, x^*)$ is a periodic orbit of Eq. (2), with period ω. From Eq. (9), it is easy to see that any state of Eq. (2), with initial condition (t_0, x^*) $(x^* \in N_{ij})$, converges to this periodic orbit exponentially as $t \to +\infty$. Hence, the isolated periodic orbit $x(t; t_0, x^*)$ located in N_{ij} is locally exponentially attractive, and N_{ij} is a locally exponentially attractive set of $x(t;t_0,x^*)$. Hence there exist n^2 isolated periodic orbits which are locally exponentially attractive. This completes the proof. □

It is worth nothing that the obtained results can be easily extended to k-neuron networks,

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + Af(x(t)) + I(t), \tag{11}$$

where $x = (x_1, x_2, \ldots, x_k)' \in \mathbb{R}^k$ denotes neuron state, $A = (a_{ij}) \in \mathbb{R}^{k\times k}$ is a real $k \times k$, each of its elements a_{ij} denotes the synaptic weights and represents the strength of the synaptic connection from neuron j to neuron i, and $I(t) = (I_1(t), I_2(t), \ldots, I_k(t))^{\mathrm{T}} \in \mathbb{R}^k$ denotes external inputs. The activation function f is a multilevel function with n segments. For any vector $x \in \mathbb{R}^k$,$f(x) = (f(x_1), f(x_2), \ldots, f(x_k))^{\mathrm{T}} \in \mathbb{R}^k$.

Theorem 3. *The k-neuron network (11) has n^k locally exponentially stable equilibrium points with constant external input I and n segments multilevel function f, if for any $i \in \{1,2,\ldots,n\}$, and $j \in \{1,2,\ldots,k\}$*

$$b_{i-1} + \sum_{l=1,l\neq j}^{k} |a_{jl}| < a_{jj}c_i + I_j < b_i - \sum_{l=1,l\neq j}^{k} |a_{jl}|.$$

Proof 3. The proof is similarly to Theorem 1. So we omit it, here. □

Theorem 4. *The neural network (11) has n^k locally exponentially limit cycles with ω-periodic input $I(t)$ and n segments multilevel function f, if for any $i \in \{1,2,\ldots,n\}$, and $j \in \{1,2,\ldots,k\}$, $t > t_0$*

$$b_{i-1} + \sum_{l=1,l\neq j}^{k} |a_{jl}| < a_{jj}c_i + I_j(t) < b_i - \sum_{l=1,l\neq j}^{k} |a_{jl}|.$$

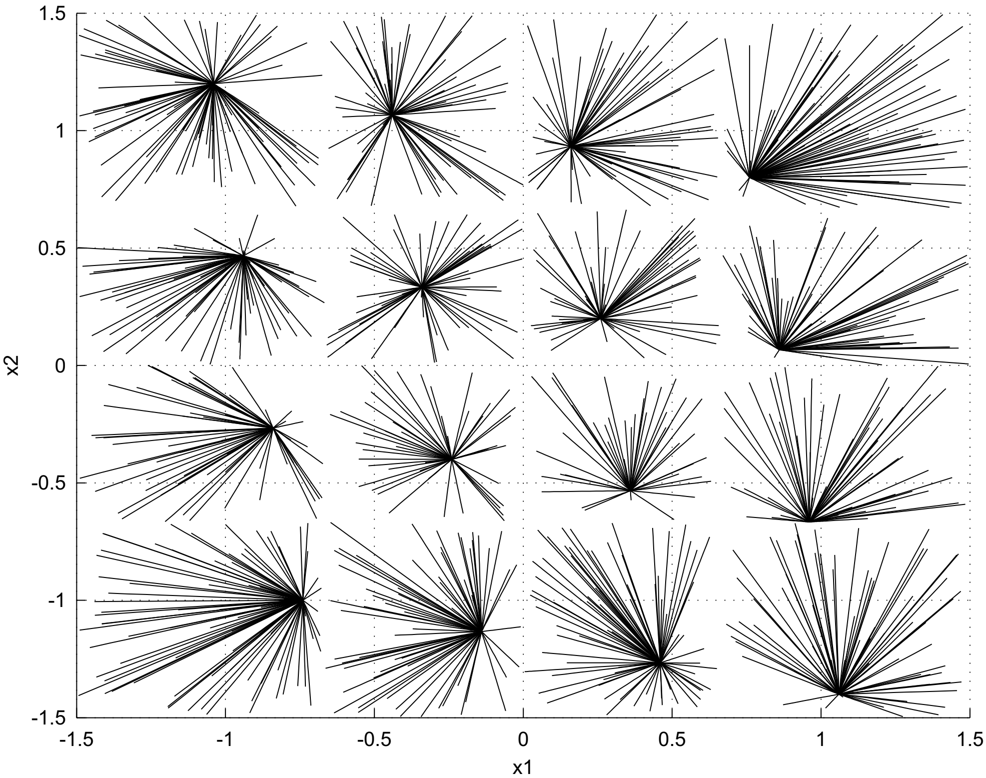


Fig. 2. Phase plot of (x_1, x_2) in Example 1.

Proof 4. The proof is similarly to Theorem 2. So we omit it, here. □

Remark. In [22], the activation functions are considered as the Fermi function:

$$g(x) = \frac{1}{1 + e^{-x/\varepsilon}}.$$

With constant external inputs, there exists 3^k equilibria under the conditions proposed in [22], and in which 2^k equilibria are stable. And in [21], with the piecewise linear activation functions, there are 2^k limit cycles, which are evoked by periodic external inputs.

From Theorem 3, by using multilevel activation function, there would be n^k stable equilibria with constant external inputs, where n is the number of segments of the multilevel activation functions. And in Theorem 4, we also proved that there exist n^k limit cycles under the periodic external inputs.

4. Three numerical examples

In this section, we give several numerical examples to illustrate the new results.

Example 1. Consider the following neural network:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = -\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0.9 & -0.15 \\ 0.2 & 1.1 \end{pmatrix}\begin{pmatrix} f(x_1(t)) \\ f(x_2(t)) \end{pmatrix} + \begin{pmatrix} 0.01 \\ -0.1 \end{pmatrix}. \tag{12}$$

where $f(x)$is a piecewise function, which has four segments.

$$f(x) = \begin{cases} -1, & x < -\frac{2}{3}; \\ -\frac{1}{3}, & -\frac{2}{3} \leqslant x < 0; \\ \frac{1}{3}, & 0 \leqslant x < \frac{2}{3}; \\ 1, & x \geqslant \frac{2}{3}. \end{cases}$$

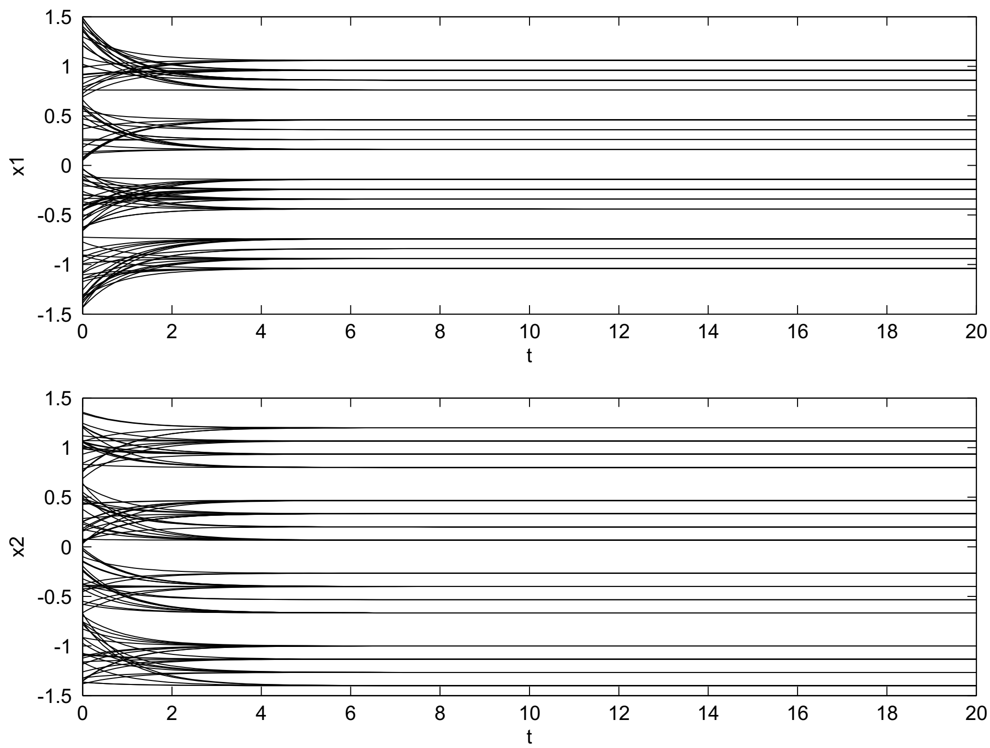


Fig. 3. Time response of x_1, x_2 in Example 1.

This network satisfies the conditions of Theorem 1. As numerical simulate shows, model (12) has $4^2 = 16$ locally stable points. (Figs. 2 and 3)

Example 2. Consider another neural network:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = -\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0.9 & -0.08 \\ 0.03 & 1.1 \end{pmatrix} \begin{pmatrix} f(x_1(t)) \\ f(x_2(t)) \end{pmatrix} + \begin{pmatrix} 0.07\sin(t) \\ 0.08\cos(t) \end{pmatrix}. \tag{13}$$

where $f(x)$ has five segments.

$$f(x) = \begin{cases} -1, & x < -\frac{3}{4}; \\ -\frac{1}{2}, & -\frac{3}{4} \leqslant x < -\frac{1}{4}; \\ 0, & -\frac{1}{4} \leqslant x < \frac{1}{4}; \\ \frac{1}{2}, & \frac{1}{4} \leqslant x < \frac{3}{4}; \\ 1, & x \geqslant \frac{3}{4}. \end{cases}$$

For network (13) the conditions of Theorem 2 hold. As numerical simulate shows, model (13) has $5^2 = 25$ locally exponentially limit cycles (Figs. 4 and 5).

Example 3. In this example, we simulate the three-dimensional neural network:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = -\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 0.95 & 0.1 & 0.1 \\ 0.2 & 1 & 0.1 \\ -0.2 & 0.1 & 1 \end{pmatrix} \begin{pmatrix} f(x_1(t)) \\ f(x_2(t)) \\ f(x_3(t)) \end{pmatrix} + \begin{pmatrix} 0.07\sin(t) \\ 0.08\cos(t) \\ 0.1(\sin(t) + \cos(t)) \end{pmatrix}. \tag{14}$$

where $f(x)$ is a multilevel function, which has three segments.

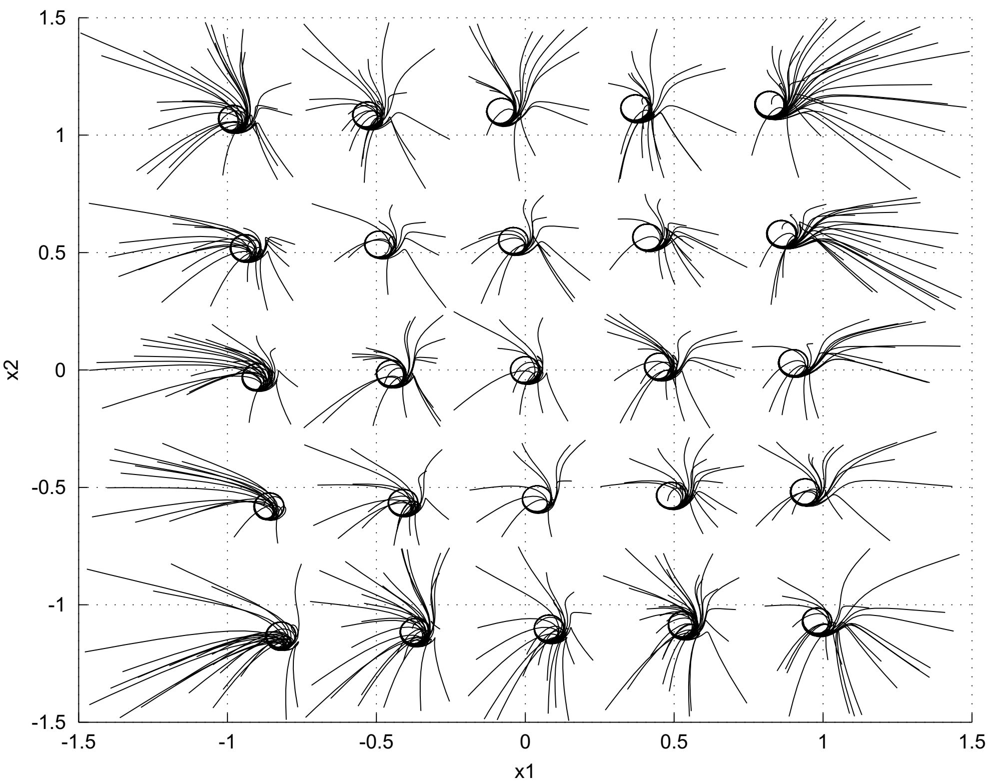


Fig. 4. Phase plot of (x_1, x_2) in Example 2.

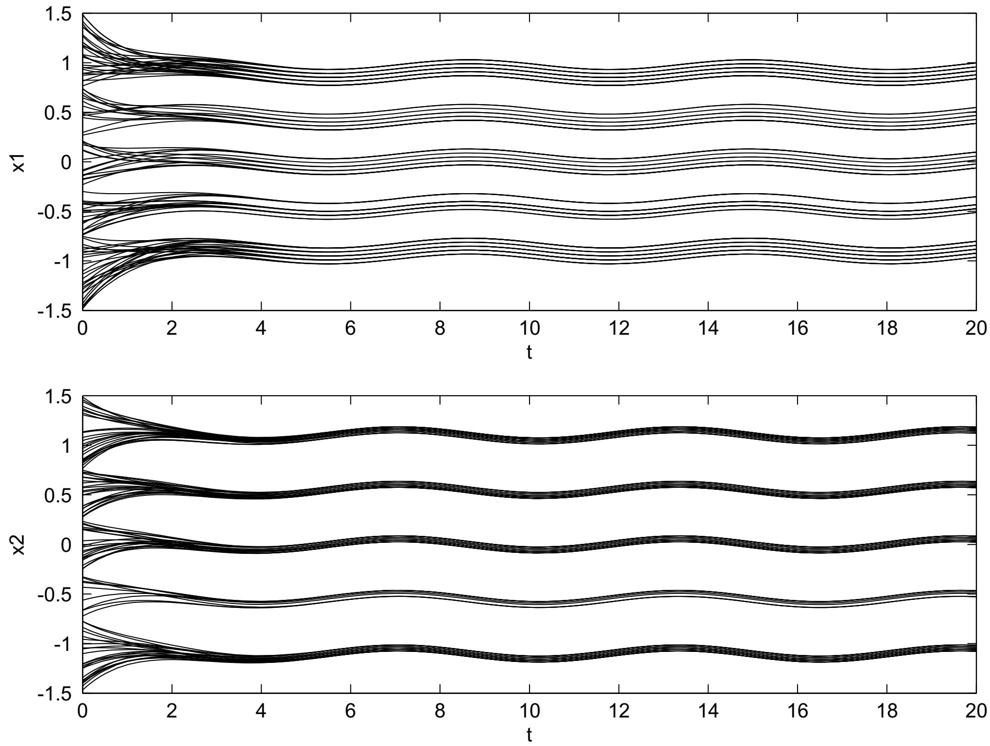


Fig. 5. Time response of x_1, x_2 in Example 2.

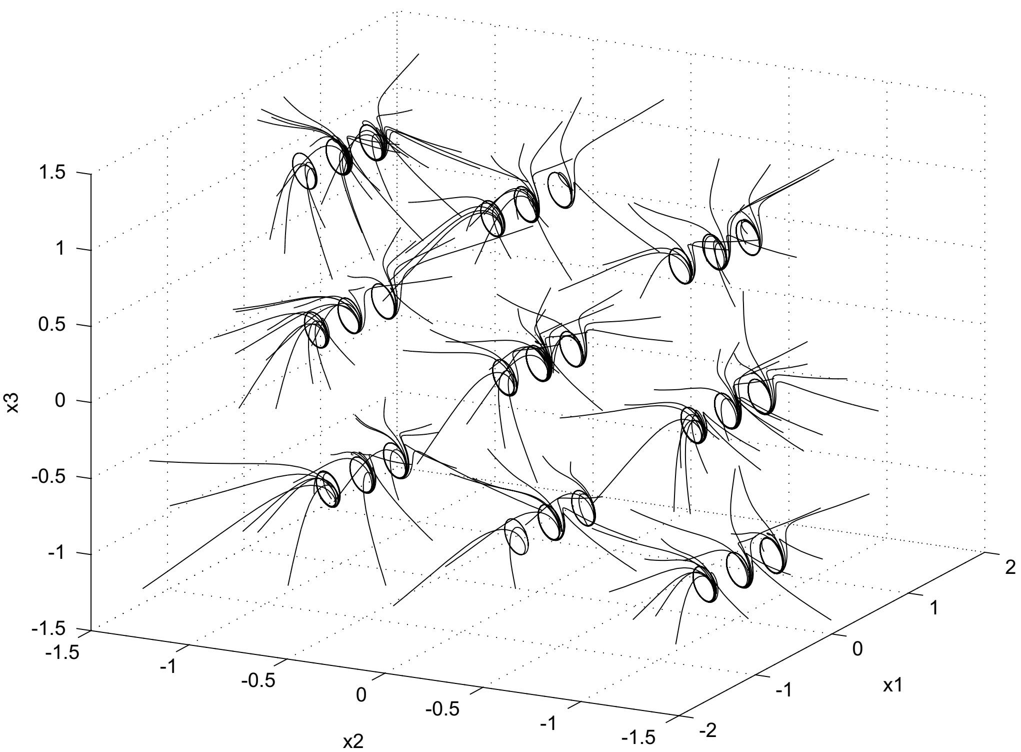


Fig. 6. Phase plot of (x_1, x_2, x_3) in Example 3.

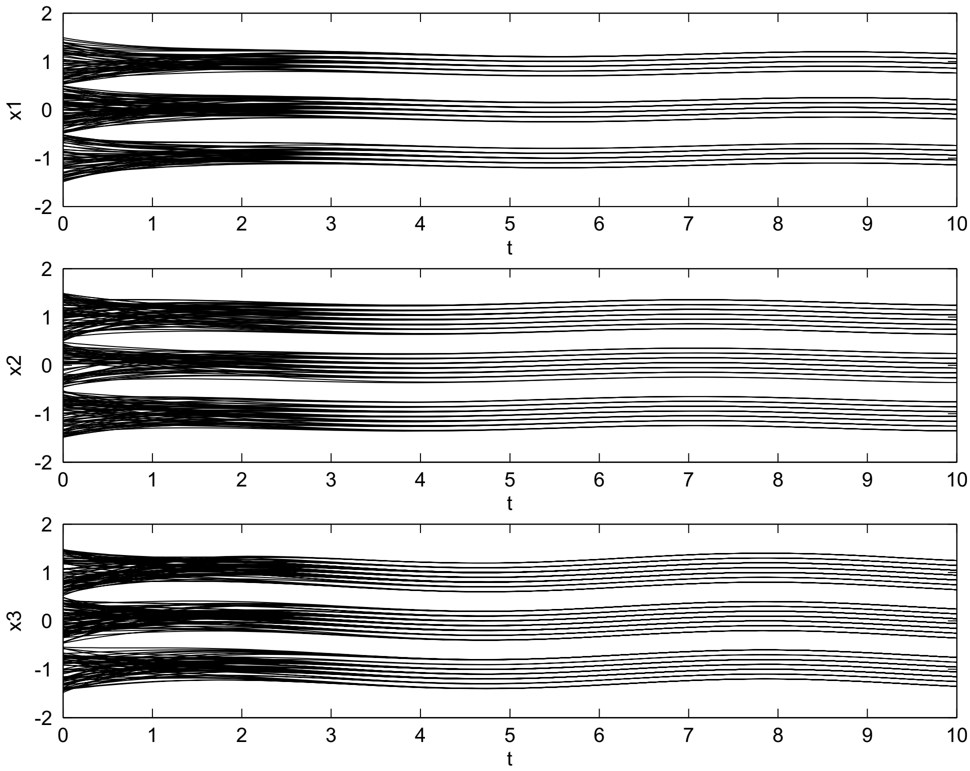


Fig. 7. Time response of x_1, x_2, x_3 in Example 3.

$$f(x) = \begin{cases} -1, & x < -\frac{1}{2}; \\ 0, & -\frac{1}{2} \leqslant x < \frac{1}{2}; \\ 1, & x \geqslant \frac{1}{2}. \end{cases}$$

This parameters also satisfy the conditions of Theorem 4. As numerical simulate shows, there are $3^3 = 27$ locally exponentially limit cycles. We demonstrate the dynamics as well as evolutions of components $x_1(t), x_2(t), x_3(t)$ for the system in Figs. 6 and 7, respectively.

5. Conclusion

In this paper, the multistability has been studied for two-dimensional neural networks with multilevel type of activation function, and we extended the obtained results to the k-neuron networks. In associative memories, as a practical application of neural networks, the capacity of memories can be arbitrary large as you need, even though the dimension of the neural network is really small. The simulation results show the characteristics of the multistability. However, the distributing of the equilibrium points is still an open problem. Also, the dynamics of the neural networks with delay is another interesting topic to be investigated in near future.

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